Ultracold Fermi gases

A/Prof Chris Vale
Swinburne University of Technology
cvale@swin.edu.au

Overview

- Three lectures covering:
  - History and development of the field
  - Ideal Fermi gases
  - Interactions in cold gases
  - Feshbach resonances
  - BEC-BCS crossover
  - Universality

- We will cover some of the basic ideas underpinning these topics

Useful resources

- Ideal Fermi gases:
  - D.V. Schroeder, "An introduction to thermal physics", Addison-Wesley (2000)
  - S. Giorgini, et al. "Ultracold Fermi gases", Rev. Mod. Phys. 80, 1215 (2008)

- Cold collisions:
  - J. Dalibard, "Collisional dynamics of ultracold atoms", Proc Int School of Physics Enrico Fermi, Course CXL: Bose -- Einstein condensation in gases, Varenna, (eds) M. Inguscio, S. Stringari, C. Wieman (1998)
  - J. Walraven, "Elements of quantum gases", http://staff.science.uva.nl/~walraven/walraven/Publications_files/Elements-of-Quantum-Gases-I.pdf (2009-10)
  - Landau and Lifshitz, "Quantum mechanics", Ch XVII, Oxford (2002)

- BEC-BCS crossover:
  - S. Giorgini, et al. "Ultracold Fermi gases" Rev. Mod. Phys. 80, 1215 (2008).
  - M. Zwierlein, PhD thesis, MIT (2006) http://cua.mit.edu/ketterle_group/Theses/theses.htm
  - W. Zwerger (ed), "BEC-BCS crossover and the unitary Fermi gas", Springer (2011)

Historical overview

- The discovery of superfluidity/superconductivity about a century ago set the early scene for a lot of what we'll cover
- In 1908 Kammerlingh Onnes liquefied He-4 down to temperatures ~ 1 K
- He noticed a discontinuity in the heat capacity below 2.2 K (later discovered to be the lambda point = critical T)
- At this point the He-4 Bose condenses and forms a superfluid
- In 1911 he used liquid He-4 to cool mercury below 4.2 K and noticed its resistance vanished discontinuously (similar results in Tin and Lead soon followed)
- This was the first demonstration of superconductivity
- Onnes won the 1913 Nobel Prize for his work on Liquid He

Historical overview

- At the time of their discovery these new behaviours were recognised to be connected to quantum physics but the theory was just in its infancy
- The fact that superfluidity in He-4 and superconductivity occur at very similar temperatures is a 'technical' coincidence but it does make one look for connections
- Superfluidity and Bose-Einstein condensation (BEC) are bosonic phenomena so why should electrons in a metal behave like a superfluid??
- The theory of what is now known as conventional superconductivity took about 50 yrs to develop and was developed by Bardeen Cooper and Schrieffer (BCS theory)
- This theory described the pairing of electrons in the presence of a Fermi sea and these (bosonic) pairs could condense and form a superfluid

Nobel Prize 1972

Historical overview

- Cooper pairs are very different to tightly bound bosons which could easily form a condensate and the connection between the two cases is still a major field of research
- Up until the 1980s, many materials were shown to superconduct and the BCS theory had great success in describing their properties
- In 1986-7, the measured critical temperatures for superconductivity jumped dramatically above the LN_2 temperature with the discovery of cuprate superconductors
- BCS theory was found to fail for these materials in some important regions of the phase diagram
- Electrons still paired but their properties depend upon complexities of the (quasi-2D) structure and the pairing and resulting superconductivity is still not completely understood

J. Georg Bednorz

K. Alexander Müller

Historical overview

- In the early 1970s, superfluidity was discovered in fermionic liquid He-3 although at much lower T than He-4 (~2 mK)
- It is more like a BCS superconductor than a Bose-Einstein condensate but it is neutral and involves p-wave pairing
- High Tc materials (unconventional superfluids) are beyond the simple BCS model
- It was proposed as early as 1950 by Fritz London that fermionic superfluidity is a pair condensate in momentum space in contrast with a BEC of tightly bound pairs in real space
- In 1980 Leggett showed that these limits should be connected smoothly connected

  - the BEC-BCS crossover

Superfluidity vs Bose condensation

- While these two phenomena are intimately connected, they are by no means equivalent
- In some cases one can occur without the other (eg. an ideal Bose gas can form a BEC but has vanishing critical velocity while a 2D Bose gas can form a superfluid but not a condensate at finite *T*)
- The common thread however, is that both a BEC and a superfluid are described by a macroscopic wavefunction

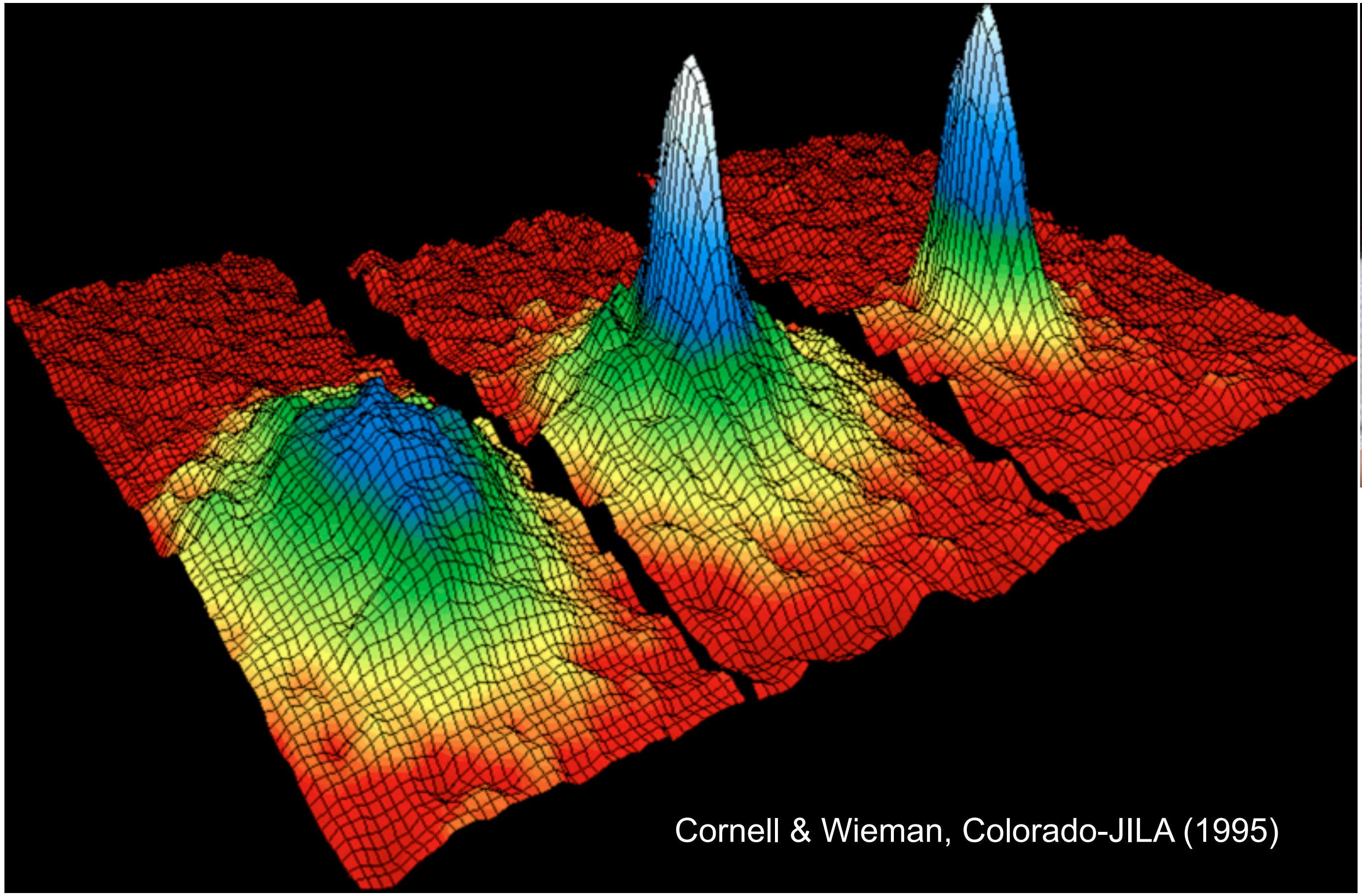
$$\Psi \propto \sqrt{n(r)}e^{i\phi(r)}$$

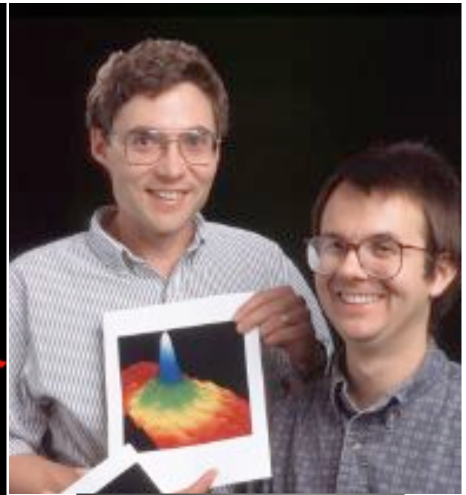
- Superfluids are characterised by flow without dissipation below a critical velocity v_c
- BEC is characterised by macroscopic occupation of a single state and occurs when the mean interparticle spacing is of the same order as the de Broglie wavelength (~ one particle per $\hbar$ of phase space)

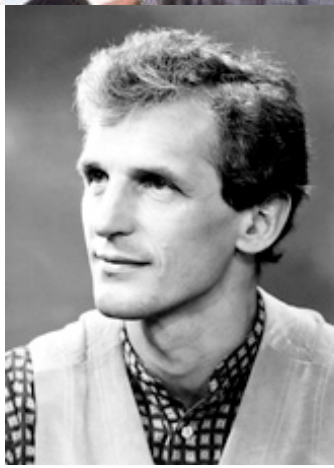
$$n\lambda_{dB}^3 \approx 1$$

Cold Atomic gases

- Atomic BECs sparked a revolution in atomic/quantum physics



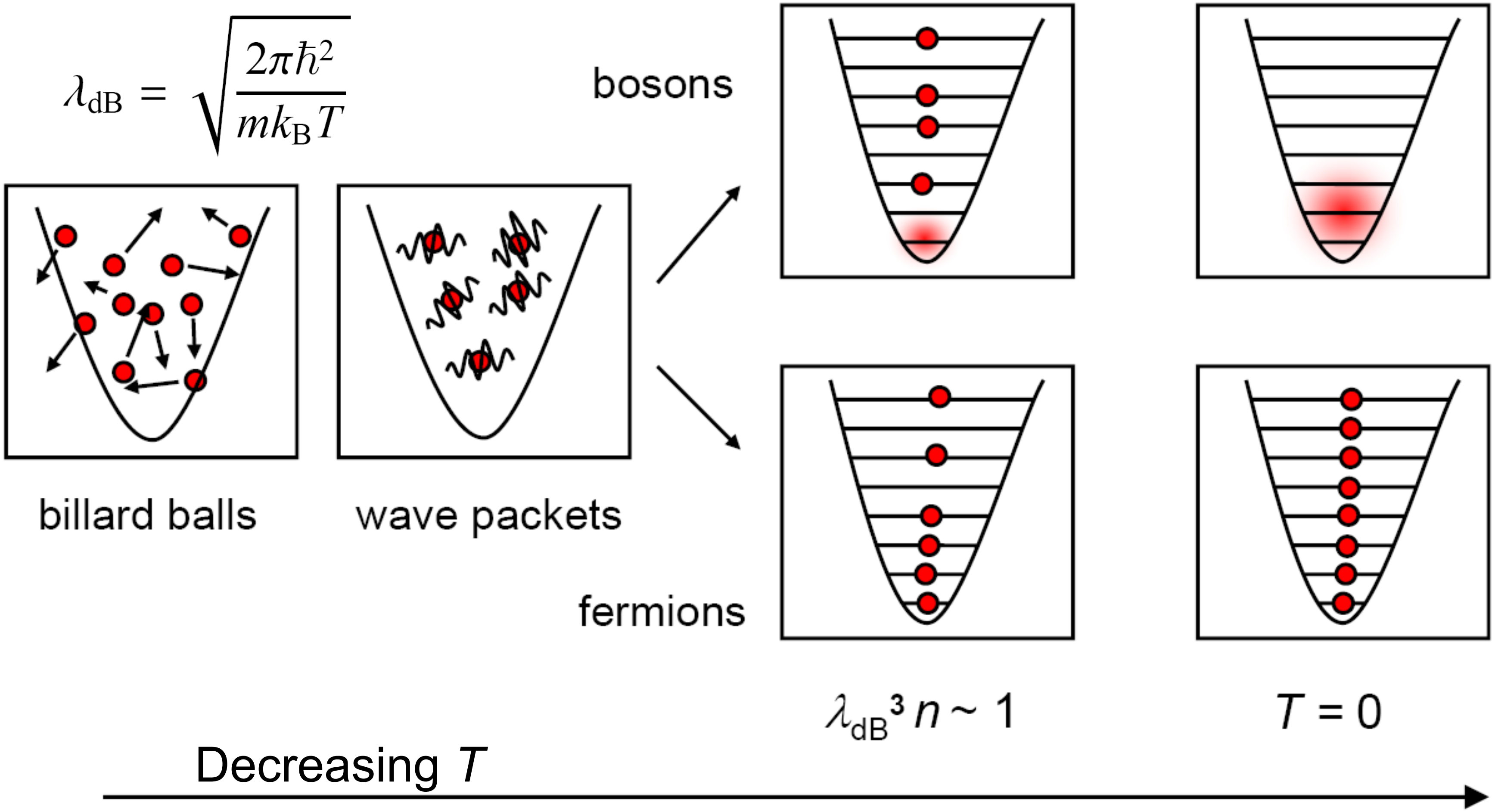




- Shortly after this, several groups began working on fermions

Cold Fermi gases

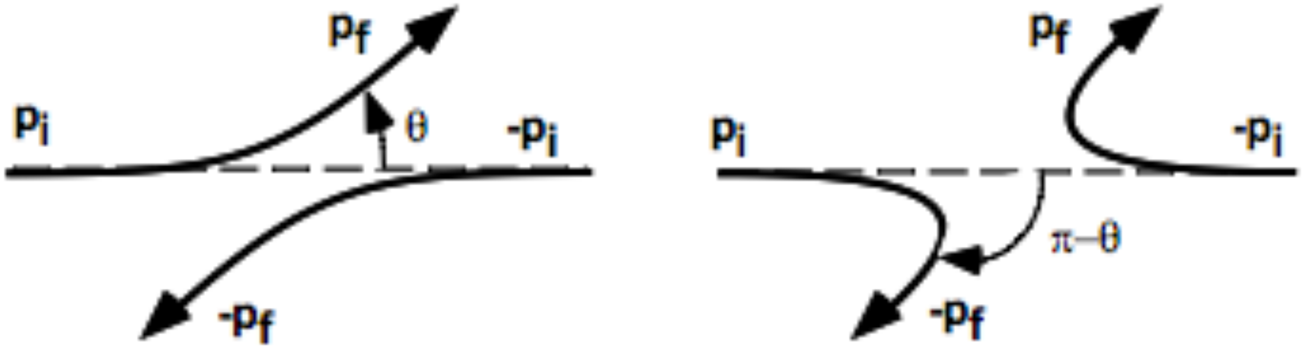
- Fermions obey the Pauli exclusion principle and show no phase transition for an ideal (non-interacting) gas



(S. Jochim, PhD, Innsbruck, 2004)

Cold Fermi gases

- Polarised (single component) Fermi gases are essentially ideal (s-wave suppressed by symmetry, higher order partial waves suppressed by the centrifugal barrier)



$$f_{asym}(\theta) = f(\theta) - f(\pi - \theta)$$

Only odd partial waves contribute

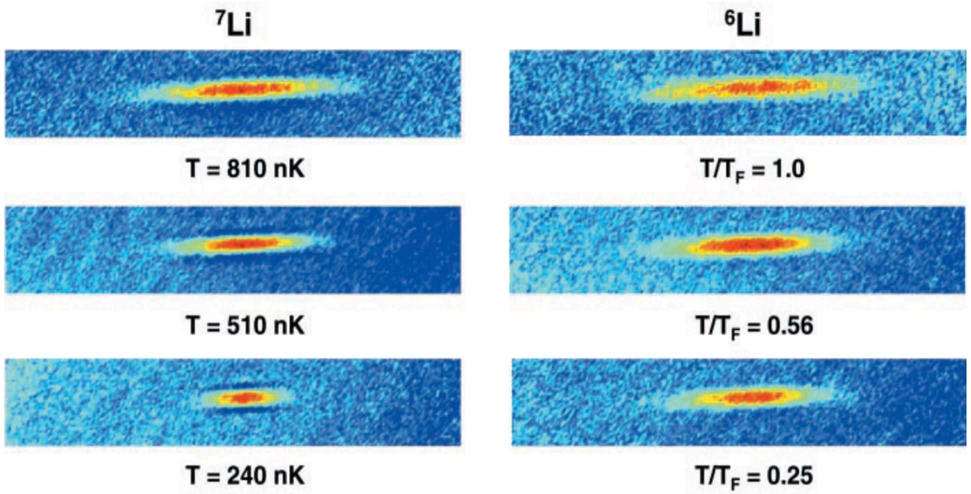
$$\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

$\sim 3\,\mathrm{mK}$

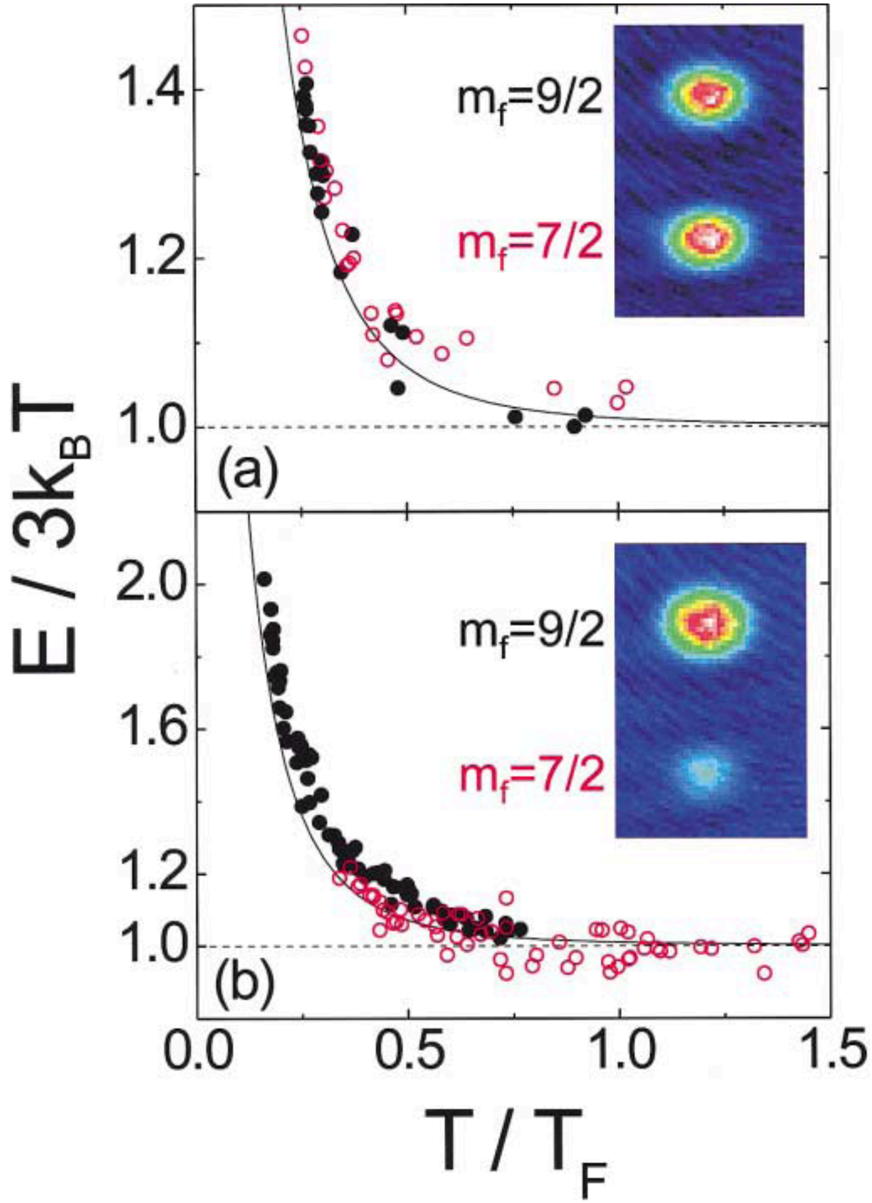
Cold Fermi gases

- Ideal Fermi gases readily display Pauli exclusion

- The energy departs from the classical value below the degeneracy temperature



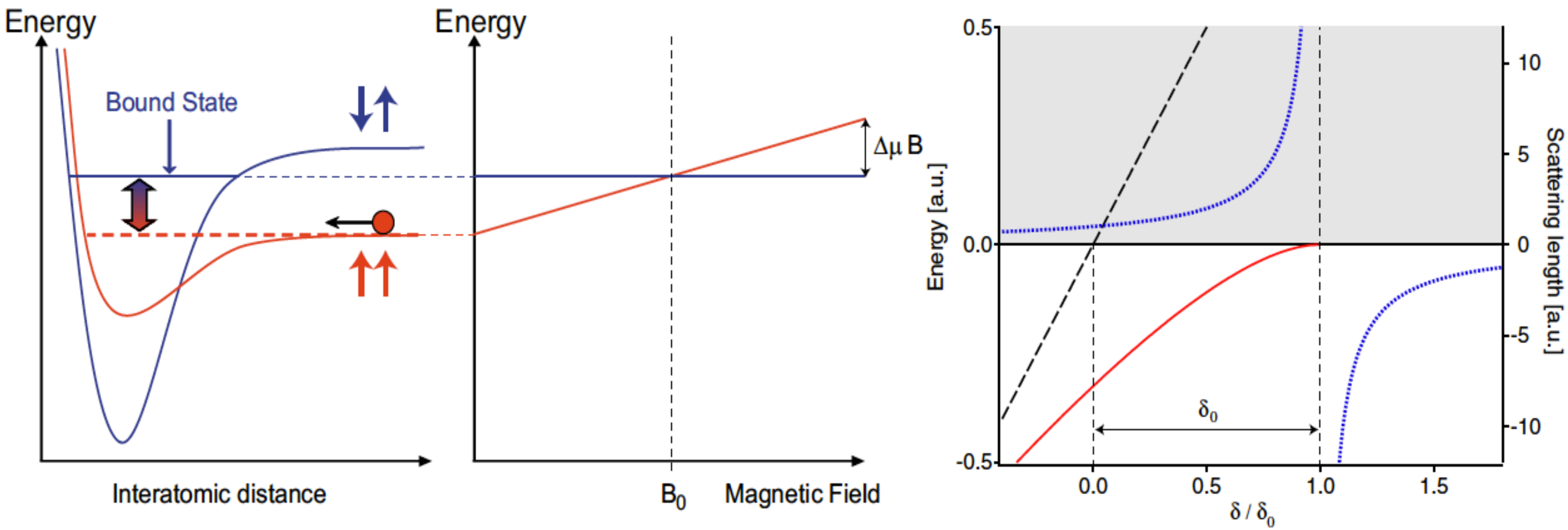
A. G. Truscott, K. E. Strecker, W. I. McAlexander, G. B. Partridge, and R. G. Hulet, Science 291, 2570 (2001).



B. DeMarco, S.B. Papp, and D.S. Jin, Phys. Rev. Lett. **86**, 5409 (2001).

Fano-Feshbach resonances

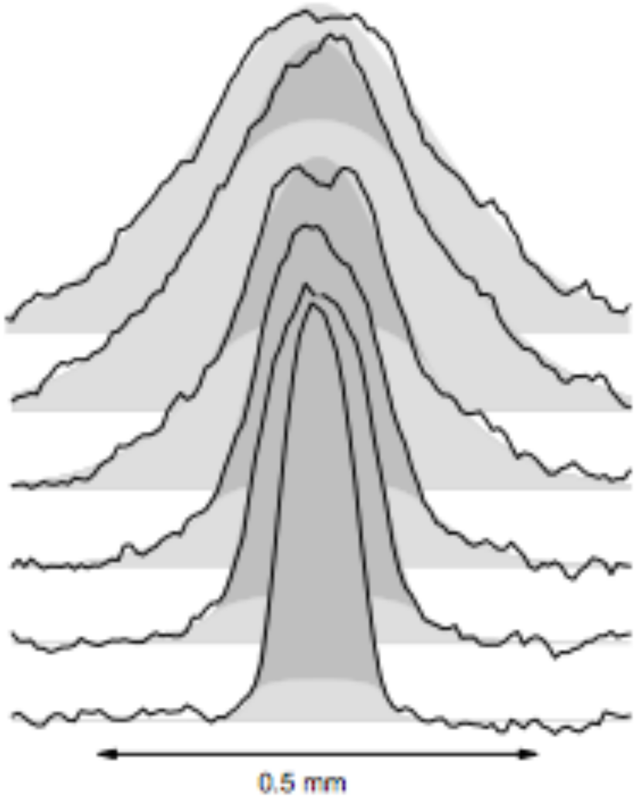
- First suggested in the context of nuclear scattering (Feshbach 1958,62)
- Ideas applied to excited atomic systems by Fano (1961)
- First observed in cold atoms in 1998 at MIT (bosons – rapid losses)
- Observation linked to formation of weakly bound molecules

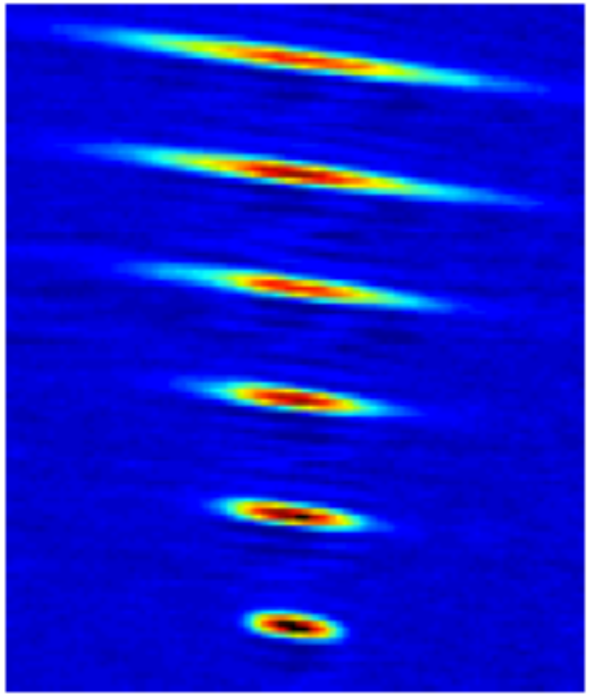


(M. Zweirlein, PhD, MIT, 2006)

Interacting Fermi gases - MBEC

- In 2003 the Innsbruck group of Rudi Grimm made a long-lived molecular BECs using $^6Li_2$ dimers



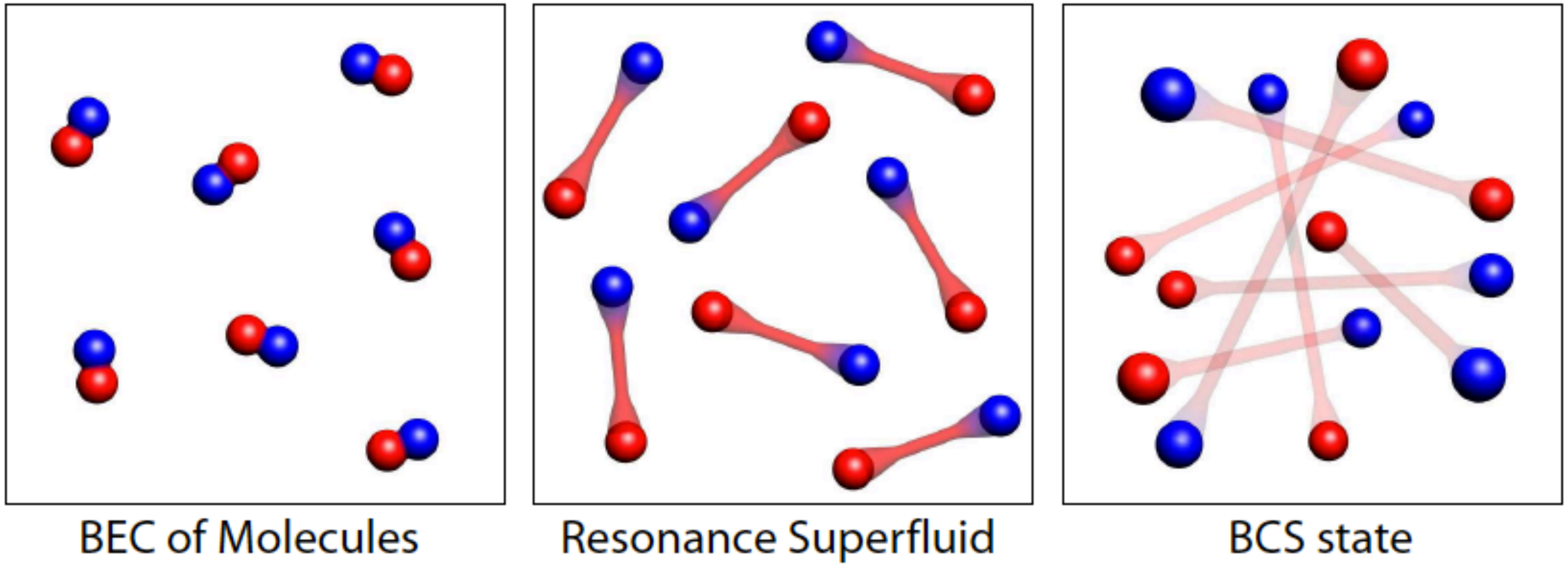


(S. Jochim, PhD, Innsbruck, 2004)

- BEC limit of the BEC-BCS crossover with interacting fermions…
- Other groups too (JILA, MIT, ENS, Rice, Duke)

BEC-BCS crossover

- Focus shifted towards studying the BEC-BCS crossover using Feshbach resonances to control interactions
- Interactions determine how pairs form and this pairing dictates the physics

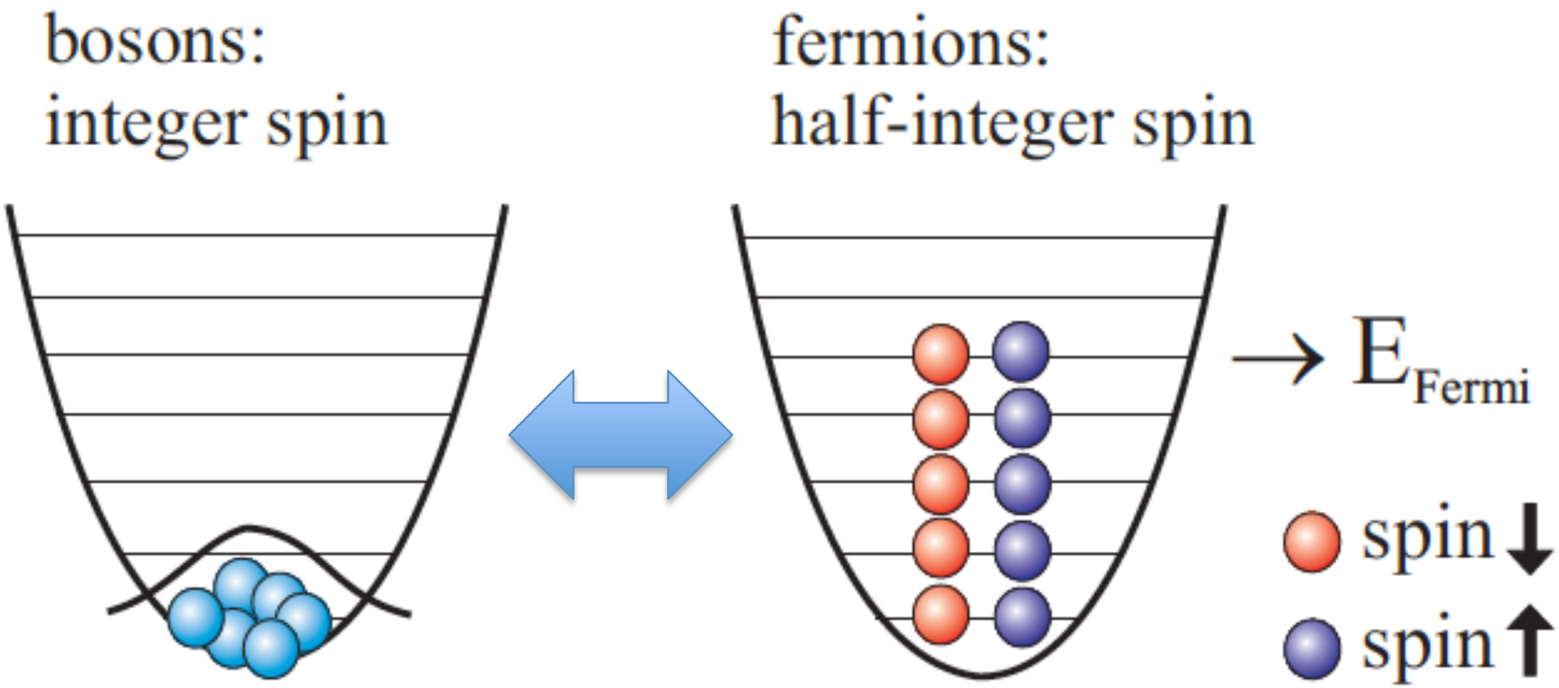


(M. Zweirlein, PhD, MIT, 2006)

- The model is a 2 component Fermi gas (spin up/ spin down) in which s-wave interactions can be tuned to span the range above

BEC-BCS crossover

- It is remarkable to find a system where the bosonic and fermionic limits are smoothly connected !?



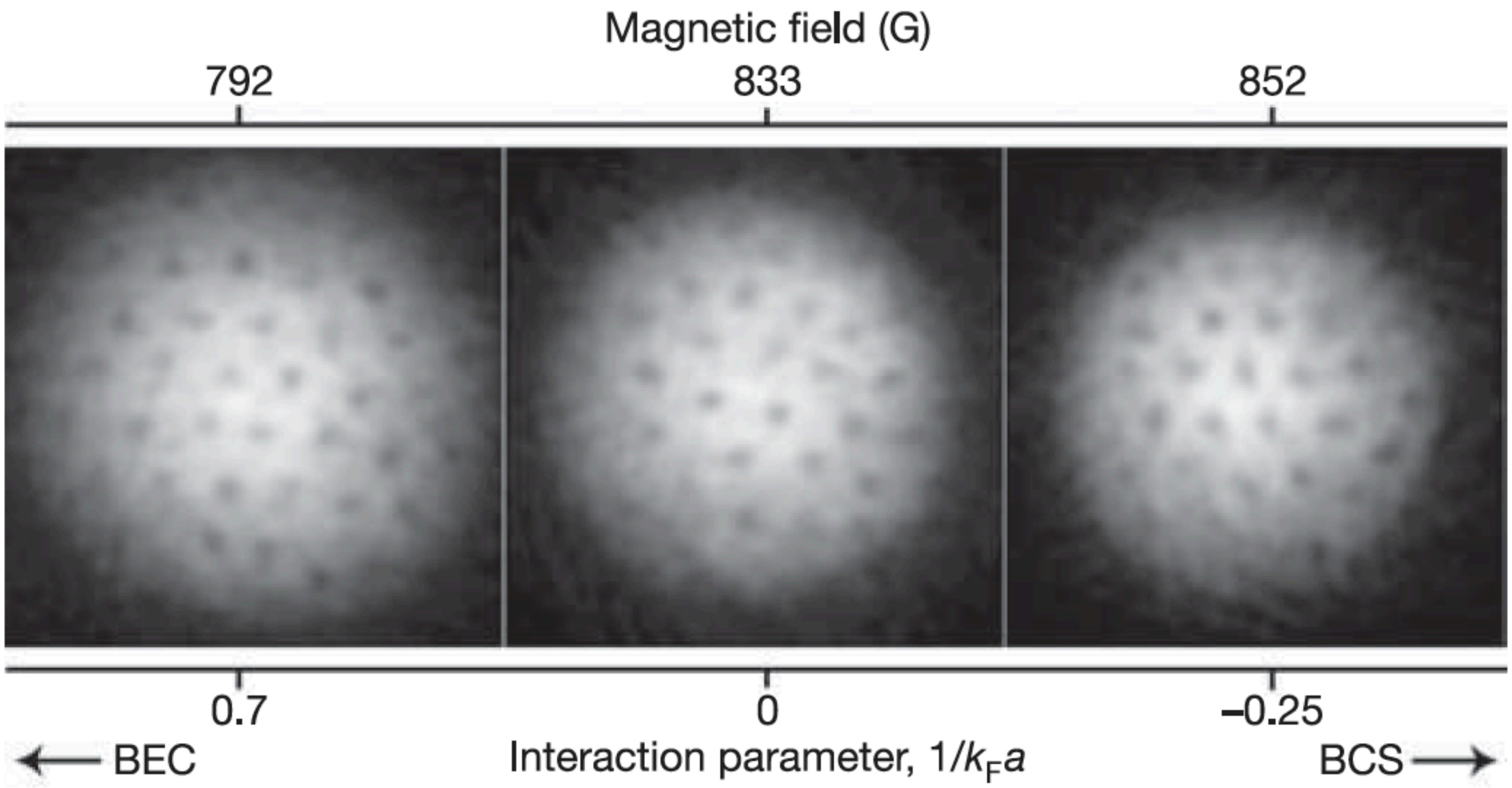
(C. Regal, PhD, Colorado - JILA, 2006)

- How can these be connected? What happens in the middle? How does a BEC evolve into a BCS superfluid?

Superfluidity in the BEC-BCS crossover

- "Smoking Gun" proof of superfluidity - vortices
- Superfluids are irrotational, angular momentum carried around vortices

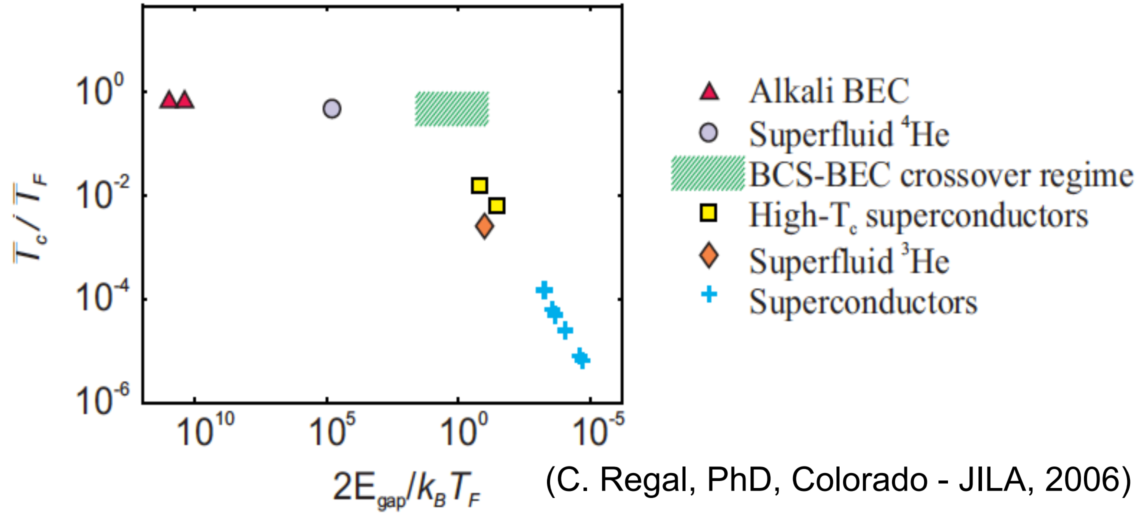
$$\mathbf{v} = \frac{\hbar}{m^*}\nabla\phi \qquad \oint \mathbf{v}\cdot d\mathbf{l} = n\frac{h}{m^*}$$



M.W. Zwierlein et al., Nature 435 , 1047-1051 (2005)

BEC-BCS crossover

- Topics of current interest:
  - Universality
  - Spin imbalanced systems
  - Higher order pairing
  - Exotic superfluids
  - Fermionic mixtures



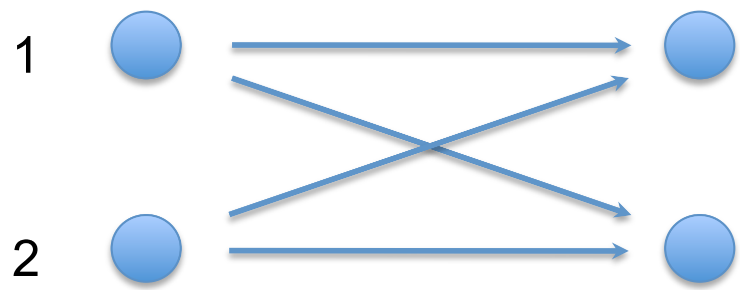
(C. Regal, PhD, Colorado - JILA, 2006)

- These topics we'll touch on throughout these lectures but next we'll have a reminder of the basics of Fermi gases before moving on to some of the more interesting topics

- We will by no means go through everything in full detail. Instead we will point to appropriate references where they exist and just go through some of the interesting points with a focus on their physical significance

Ideal Fermi gas

- We have already introduced some of the basic ideas but now we'll do some more detailed revision to provide a reminder of the key physics and provide a complete picture
- The wavefunction for two fermions (½ integer spin particles) must be anti-symmetric



$$\lim_{b \to a} \psi(a,b) = \frac{1}{\sqrt{2}}[\psi_{1,2}(a,a) - \psi_{2,1}(a,a)] = 0$$

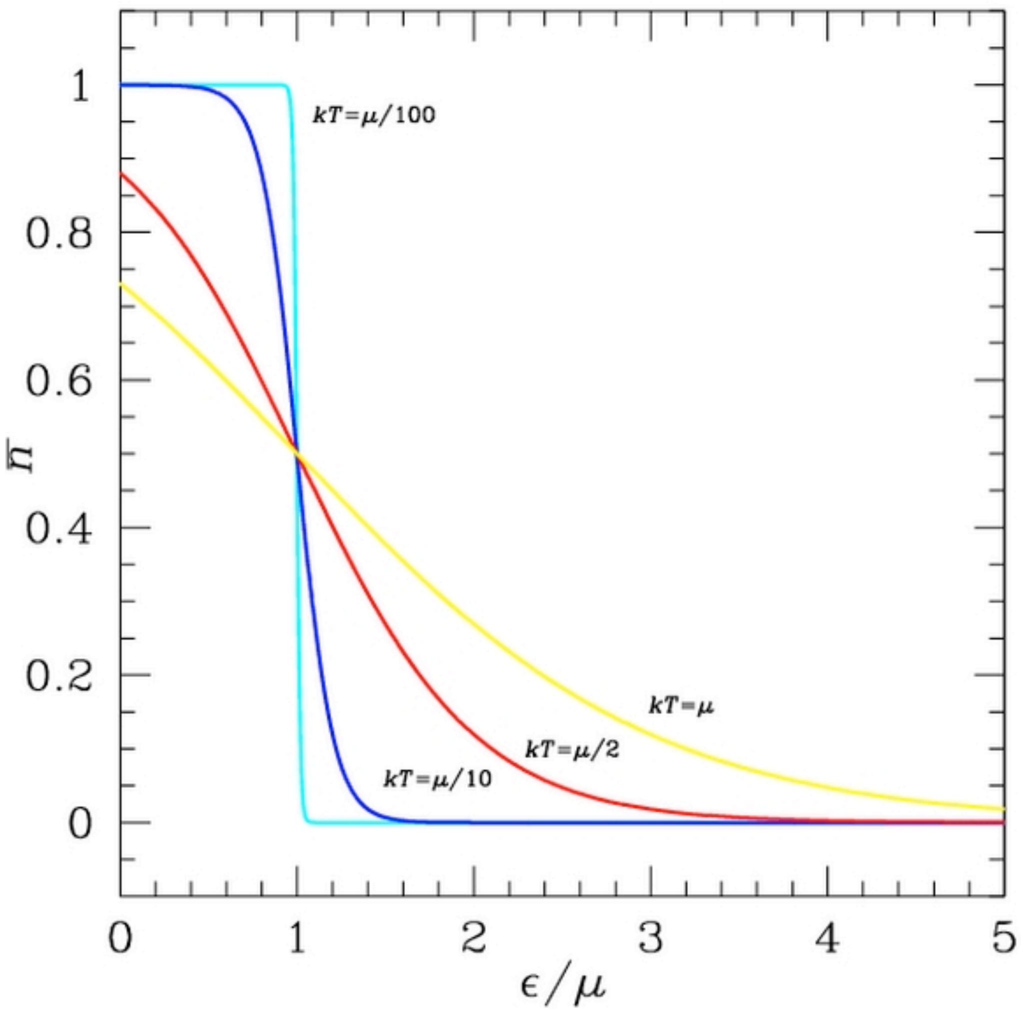
Ideal Fermi gas

- Recall the Fermi-Dirac distribution (can be found by counting the ways indistinguishable particles can fill the available states, with only one particle per state)

$$n(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_B T} + 1}$$

- At T = 0 this is a step function with a chemical potential equal to the Fermi energy (highest occupied state)

$$\mu = \left(\frac{dE}{dN}\right)_{S,V} = E_F \text{ (at } T = 0)$$



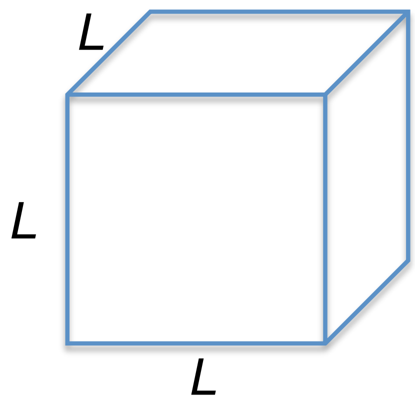
Ideal Fermi gas (homogeneous)

- In the thermodynamic limit (canonical ensemble) we have for the number of particles and total energy

$$N = \sum_i n_i = \int g(\epsilon) n(\epsilon) d\epsilon$$
$$E = \sum_i n_i \epsilon_i = \int \epsilon g(\epsilon) n(\epsilon) d\epsilon$$

where we have introduced the density of states $g(\epsilon)$

- This depends on the confining potential (container) of the gas
- Let's choose a box with side lengths *L*



Ideal Fermi gas (homogeneous)

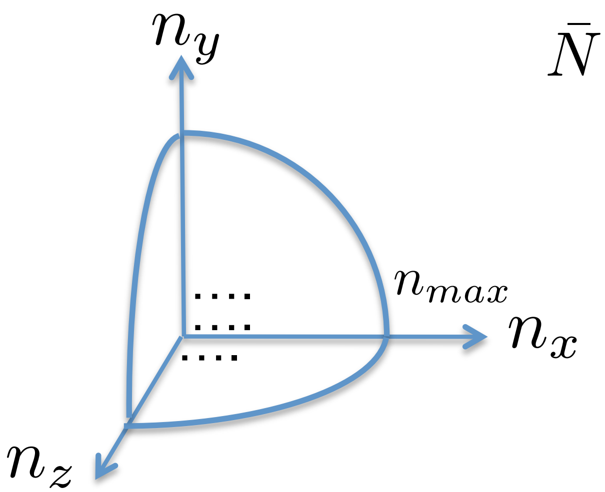
- The allowed states have momentum (integer *n*)

$$p_n = \frac{h}{\lambda_n} = \frac{hn}{2L}$$

- The allowed energies are then

$$E_n = \frac{\mathbf{p}^2}{2m} = \frac{h^2}{8mL^2}(n_x^2 + n_y^2 + n_z^2) = \frac{h^2 n^2}{8mL^2}$$

- The number of states available below a given energy is just the volume of a 1/8th sphere



$$\bar{N}(E) = \frac{1}{8}.\frac{4}{3}\pi n^3 = \frac{4\pi L^3}{3}\left(\frac{2mE}{h^2}\right)^{3/2}$$

$$E_F = \frac{h^2 n_{max}^2}{8mL^2}$$

$$N = \frac{\pi n_{max}^3}{6}$$

Ideal Fermi gas (homogeneous)

- We can now differentiate this to obtain the density of states

$$g(\epsilon) = \frac{d\bar{N}(\epsilon)}{d\epsilon} = \frac{2\pi(2m)^{3/2}L^3}{h^3}\sqrt{\epsilon}$$

- This plays an important role in the physics we'll come across but for now lets look at what this means for the number of particles and the energy

$$N = \frac{2\pi(2m)^{3/2}L^3}{h^3}\int \frac{\sqrt{\epsilon}}{e^{(\epsilon-\mu)/k_B T}+1}d\epsilon$$

$$E = \frac{2\pi(2m)^{3/2}L^3}{h^3}\int \frac{\epsilon^{3/2}}{e^{(\epsilon-\mu)/k_B T}+1}d\epsilon$$

- The chemical potential is usually found from the first integral which is generally not analytic

Ideal Fermi gas (homogeneous)

- At T = 0, $n(\varepsilon)$ is a step function that can be integrated up to E_F

$$N = \int_0^{E_F} g(\epsilon)d\epsilon = \frac{4\pi(2m)^{3/2}}{3h^3} V E_F^{3/2}$$

$$E = \int_0^{E_F} \epsilon g(\epsilon)d\epsilon = \frac{4\pi(2m)^{3/2}}{5h^3} V E_F^{5/2}$$

- Therefore the mean energy per particle is: $\frac{E}{N} = \frac{3}{5}E_F$

- E_F depends on the density: $E_F = \frac{h^2}{2m}\left(\frac{3n}{4\pi}\right)^{2/3}$ $\qquad n = \frac{N}{V}$

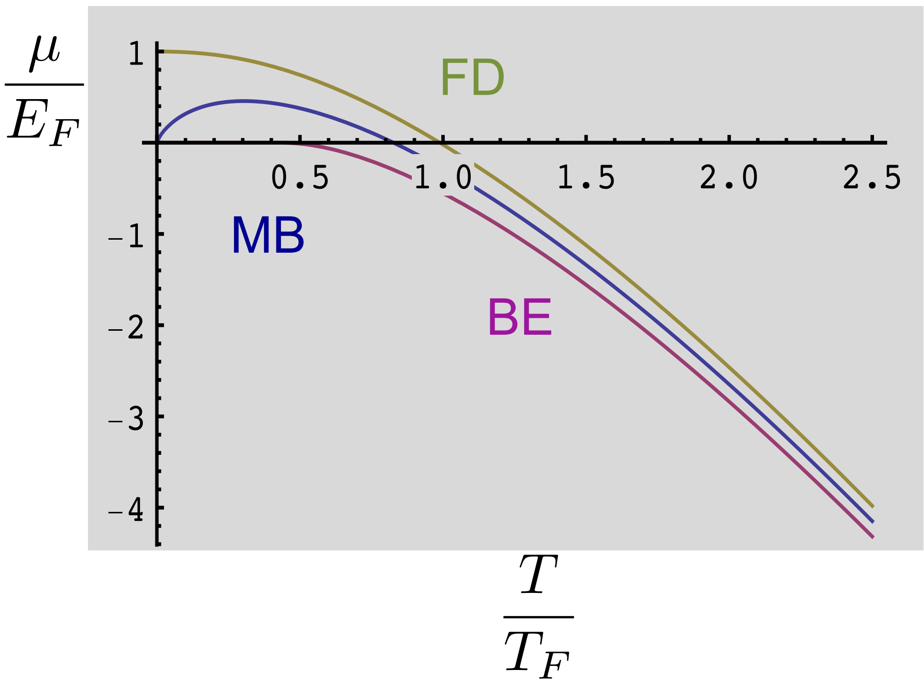
- And there is pressure… $P_{deg} = -\left(\frac{\partial E}{\partial V}\right)_{S,V} = \frac{2NE_F}{5V} = \frac{2U}{3V}$

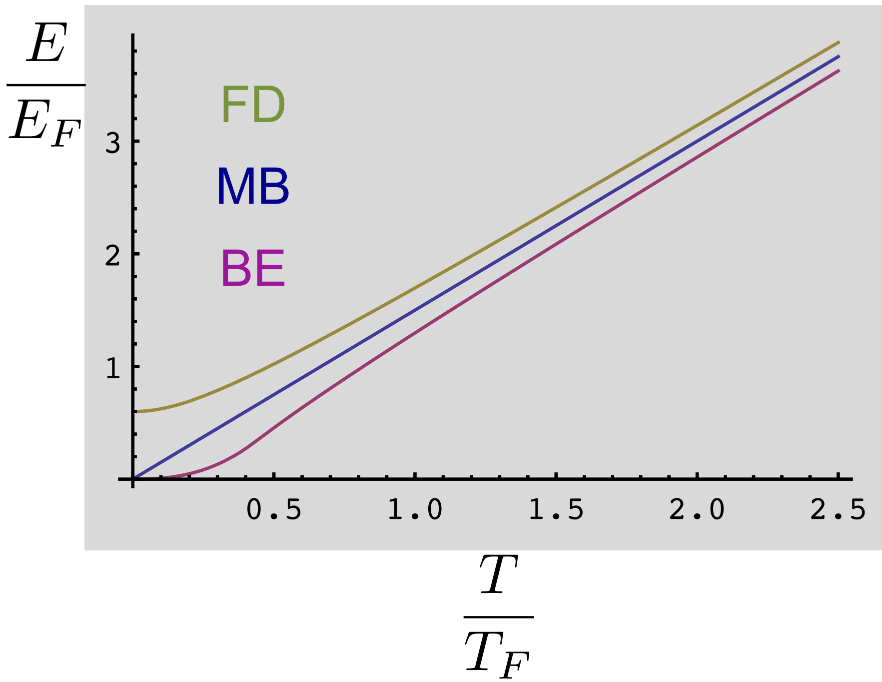
Ideal Fermi gas (homogeneous)

- At finite temperatures these integrals involve polylogarithm functions (which we'll go through soon)

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}$$

- Here are some of the results you will find for the chemical potential and energy for the classical (MB), Bose (BE) and Fermi (FD) cases





Ideal Fermi gas (trapped)

- The density of states can be found for any trapping potential using

$$g(\epsilon) = \frac{1}{h^3} \int \delta\left(\epsilon - H(r,p)\right) d^3p\, d^3r$$

- Where *H* is the classical hamiltonian: $H(r,p) = \frac{p^2}{2m} + V(\bar{r})$

- The integral over *p* is straightforward in spherical coordinates giving

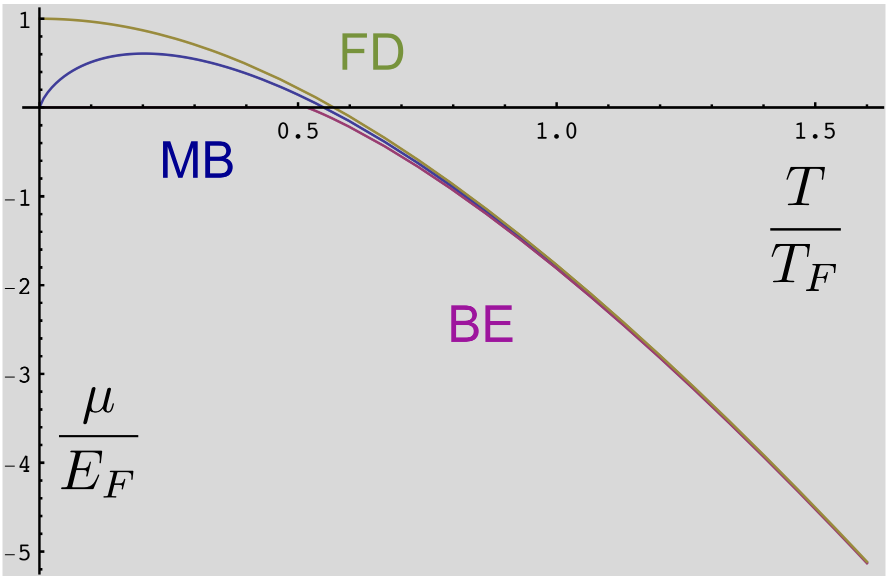
$$g(\epsilon) = \frac{2\pi(2m)^{3/2}}{h^3} \int_{\mathrm{Vol}(\epsilon)} \sqrt{\epsilon - V(\bar{r})}\, d^3r$$

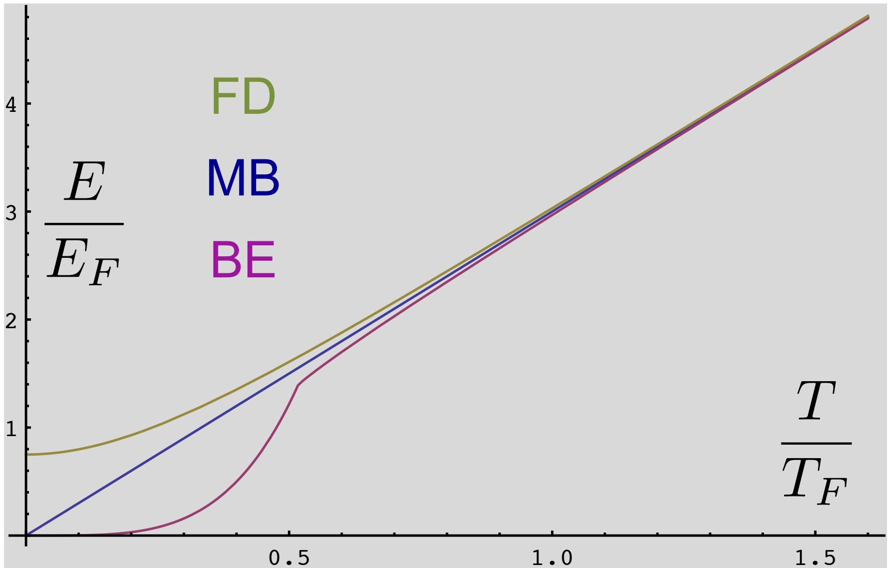
- In the case of a harmonic trap this gives:

$$g(\epsilon) = \frac{\epsilon}{2(\hbar\bar{\omega})^3} \quad \text{where} \quad \bar{\omega} = (\omega_x \omega_y \omega_z)$$

Ideal Fermi gas (trapped)

- The *E* and *N* integrals need to be evaluated numerically (this can be done in a number of ways, e.g. bisection algorithm)
- The chemical potential and total energy appear similar to the homogeneous case but quantitatively the results are quite different
- Trapped Bose and Fermi gases approach the ideal gas behaviour at much lower relative temperatures





Ultracold collisions

- The ideal Fermi and Bose gases are fairly easy to treat and many thermodynamic properties can be found exactly

- When interactions are present however, this is no longer true

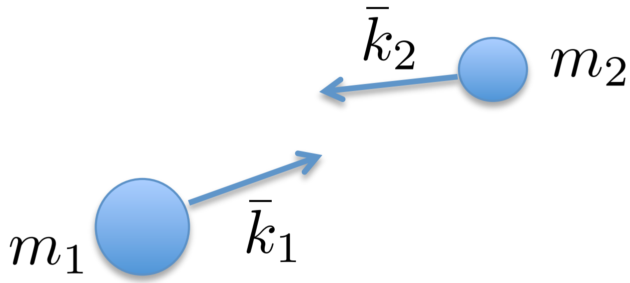
- You will have already encountered the Gross-Pitaevskii equation to describe interacting BECs yet this is an approximation which breaks down for very strong interactions

- Fermi gases can be made to interact very strongly through s-wave collisions near a Feshbach resonance

- We will now review some basic scattering theory and apply it to cold atom collisions and Feshbach resonances

Ultracold collisions

- In cold atom collisions we usually have an isotropic interaction that depends only on the distance between the two atoms



$$V(\bar{r}_1 - \bar{r}_2) = V(r)$$

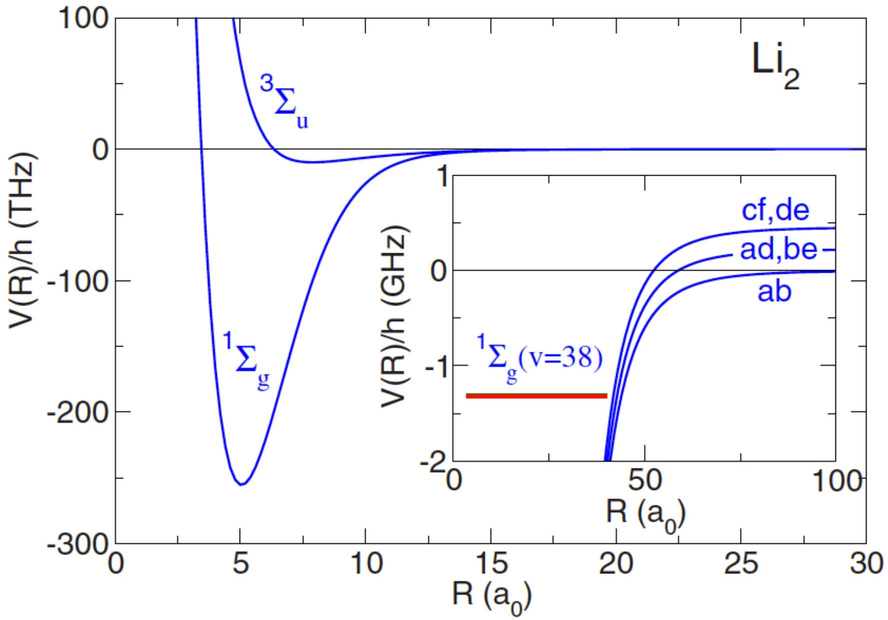
- This allows us to work in the centre of mass frame and treat the collision as a single particle with reduced mass *m* scattering from the potential *V*(*r*) with momentum *ħk* where

$$m = \frac{m_1 m_2}{m_1 + m_2}$$

$$k = |\bar{k}_1 - \bar{k}_2|$$

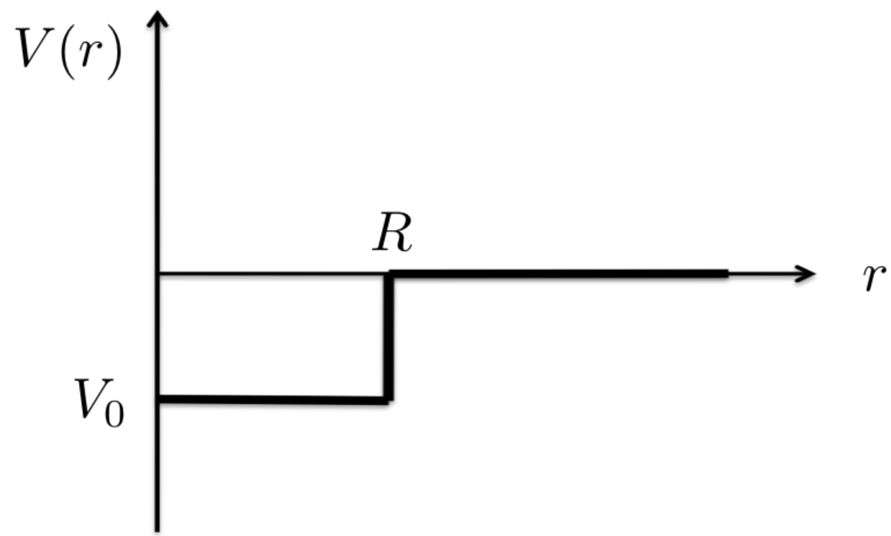
Ultracold collisions

- The interaction between atoms is set by the van der Waals potential
- In the case of Li this is shown on the left below for the triplet and singlet cases
- Note the short range of the potential (~ $50a_0$) which is much shorter than λ_{dB} at the ultracold temperatures
- This greatly simplifies the problem…



C. Chin et al., Rev. Mod. Phys. 82, 1225 (2010)

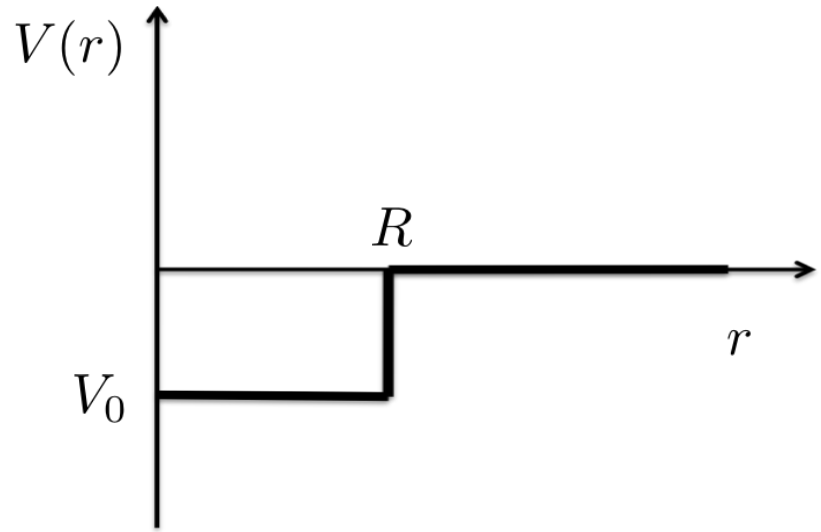
$\approx$



Ultracold collisions

- All the key physics needed to understand cold collisions and Feshbach resonances can be gleamed from the finite square (spherical) well…

- Consider the case shown to the right,
- The only parameters that matter are the depth and size of the well
- Remember this is a 3D problem where



$$V(\bar{r}_1 - \bar{r}_2) = V(r)$$

- Therefore we only need the radial Schroedinger equation for $u = r\psi$

$$\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

- Question: What must happen to *u* as $r \longrightarrow 0$

Ultracold collisions

- We won't go through the full formalism of scattering theory but will just quote some of the main results
- The details can be found in many QM texts (eg Griffiths)
- When scattering from a spherical potential we generally need to find solutions of the Schroedinger equation of the form

$$\psi(r,\theta) = A[e^{ikz} + f(\theta)\frac{e^{ikr}}{r}] \quad \text{for} \quad r \gg R$$

- That is an incoming plane wave scatters to an outgoing spherical wave with some dependence on the scattering angle described by $f(\theta)$
- This can be approached using the method of partial waves and the scattering of each partial wave calculated separately

$$f(\theta) = \sum_{l=0}^{\infty}(2l+1)a_l P_l(\cos\theta)$$

- Where l is the angular momentum, a_l is the partial wave amplitude and P_l is a Legendre polynomial

Ultracold collisions

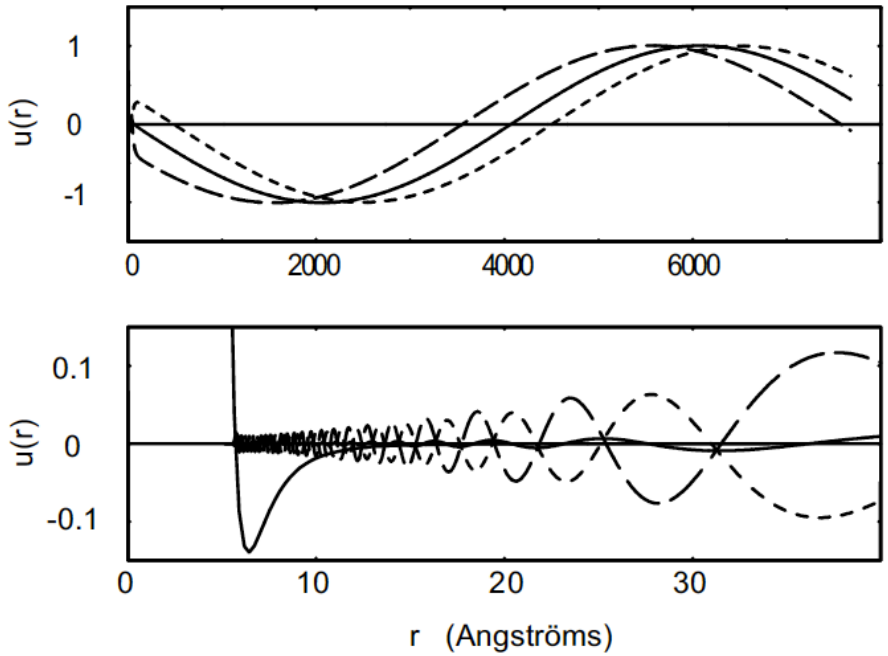
- The scattering amplitude can also be expressed in terms of a phase shift δ_l for the relevant partial wave

$$f(\theta) = \frac{1}{k}\sum_{l=0}^{\infty}(2l+1)e^{ik\delta_l}\sin(\delta_l)P_l(\cos\theta)$$

and the total scattering cross-section can be expressed as

$$\sigma = \int |f(\theta)|^2 d\Omega = \frac{4\pi}{k^2}\sum_{l=0}^{\infty}(2l+1)\sin^2(\delta_l)$$

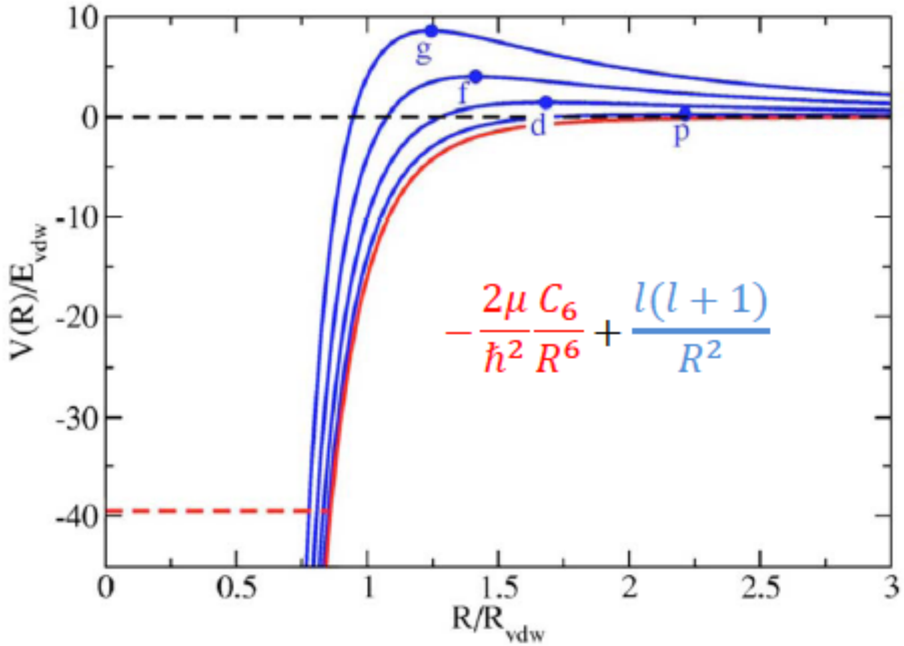
- At large *r* (>> *R*) δ_l is just the phase shift of a plane wave
- For spherical potentials, the amplitude of each partial wave is conserved



Ultracold collisions

- At low energy (low k) only the *l* = 0 partial wave contributes… why??

- Conceptually this is because low energy particles can't climb over the centrifugal barrier



- More precisely it can be shown

$$\delta_l \propto k^{2l+1}$$

which obviously vanishes quickly for *l* > 0 at small *k*

- Introducing the s-wave scattering length, defined by

$$a = -\lim_{k\to 0}\frac{\tan\delta_0}{k}$$

- We can rewrite $f_0(k)$ for k→0 as:

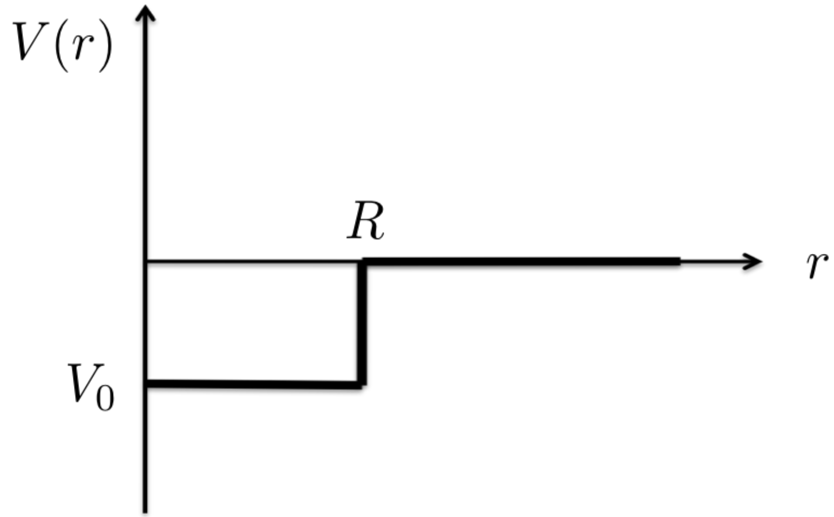
$$f_0 = \frac{1}{k\cot\delta_0 - ik} = \frac{-a}{1+ika}$$

Ultracold collisions

- So now back to the square well…
- In 3D we can easily show that the solution at large *r* is

$$u_0(r) = A\frac{e^{i\delta_0}}{k}\sin\left(kr+\delta_0\right)$$

- With $k \ll 1/R$, we do not resolve the fine details of the potential



- This means λ_{dB} is large so δ_0 is also small which means (c.f. hard sphere)

$$\tan\delta_0 \approx \delta_0 \quad \therefore \quad \delta_0 \approx -ka$$

- The solution outside and inside the well is then

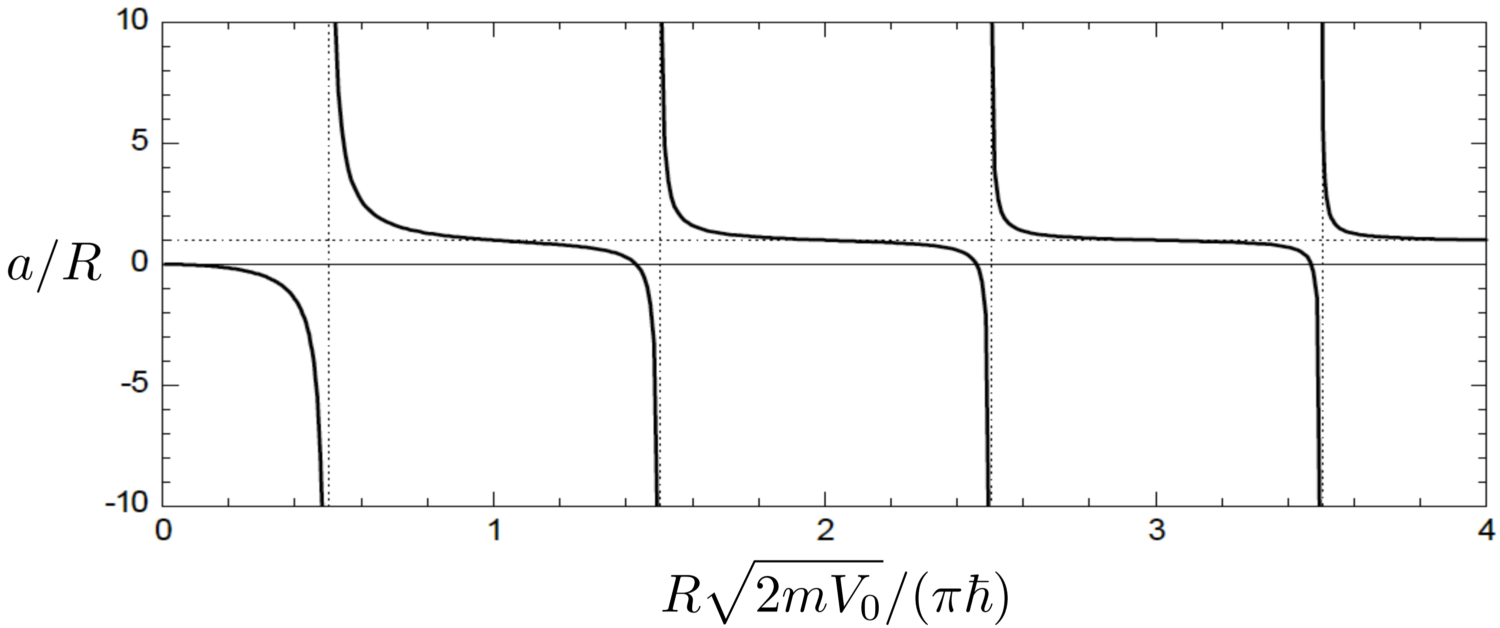
$$u(r) = \begin{cases} A\sin\left(k'r\right) & r \leq R \\ B(r-a) & r > R \end{cases} \qquad \text{where} \quad k' = \sqrt{2mV_0}/\hbar$$

- This can now be solved by continuity of <u>*u*</u> and *u*' at *r* = *R*

Ultracold collisions

- We find – $A\sin(k'R) = B(r-a)$ and $Ak'\cos(k'R) = B$

- This gives the scattering length as $a = R\left[1 - \frac{\tan(k'R)}{k'R}\right]$



- The scattering length changes dramatically as a function of well depth
- It diverges and repeats periodically (same at small and large V_0)
- What's going on??

Ultracold collisions

- The scattering length, a, blows up when

$$k'R = \pi/2 \quad \text{or} \quad V_0 = V_c = \frac{\pi^2\hbar}{8mR^2};$$

- At this point $u' \to 0$ and $\psi = u/r$ now decreases at $r \gg R$
- When the depth of the potential exceeds the critical value V_c, it can support a bound state…
- For $E < 0$ we expect:

$$u(r) = \begin{cases} A\sin(k'r) & r \le R \\ Ce^{-\kappa r} & r > R \end{cases}$$

- Using continuity of u and u' again we find

$$\kappa = -k' \cot(k'R)$$

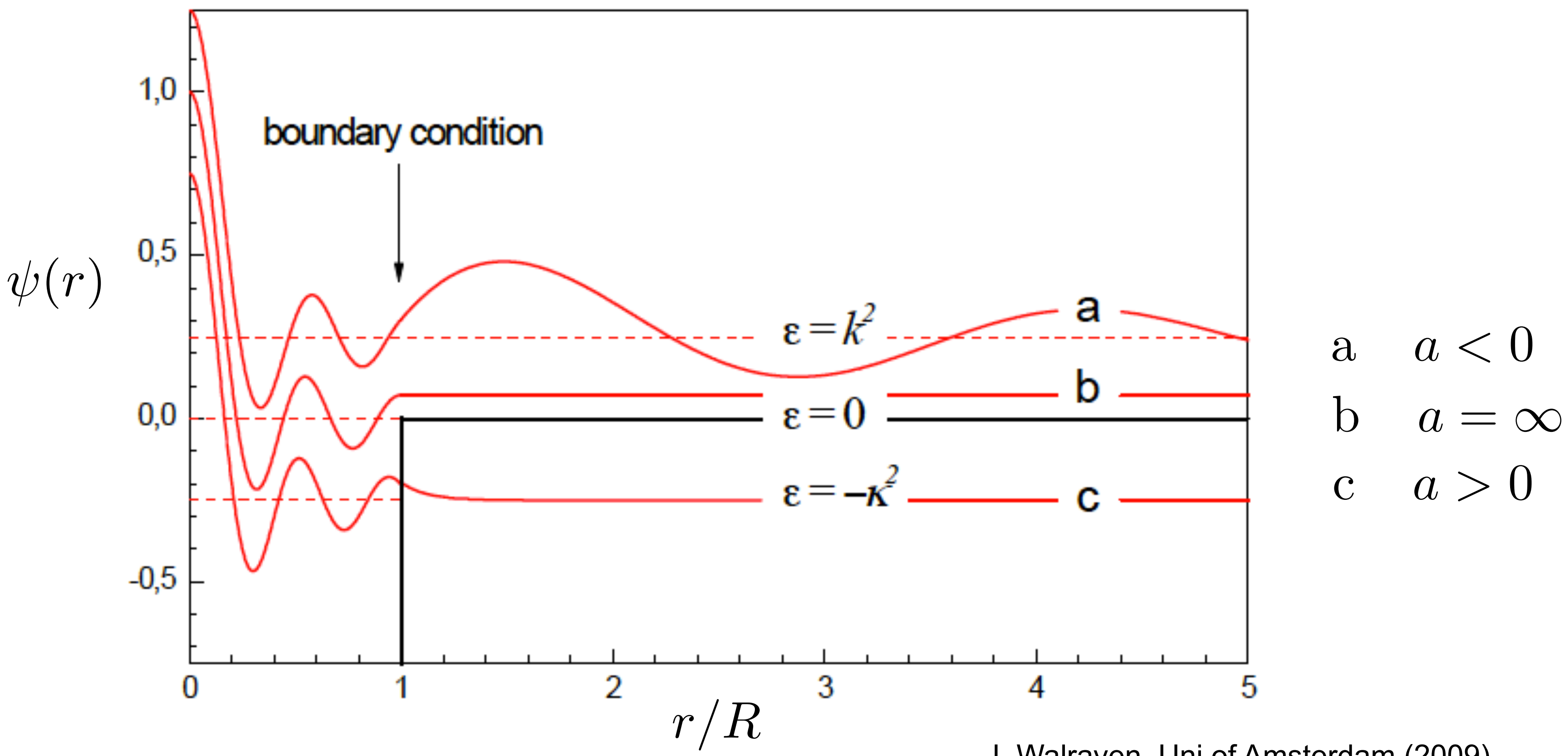
- Which gives at large a: $\kappa \approx \frac{1}{a}$ What does this mean physically??

Bound states

- We can use this to estimate the binding energy

$$E_B = -\frac{\hbar^2 \kappa^2}{2m} \approx -\frac{\hbar^2}{2ma^2} \quad \text{for} \quad a \gg R$$

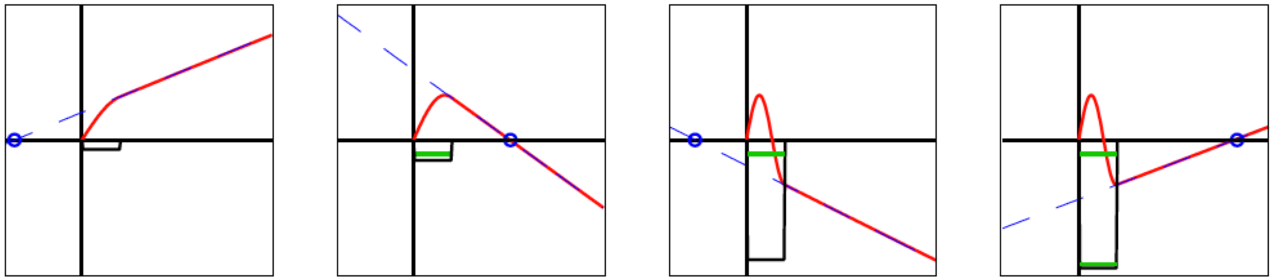
- This can now be visualised for $\psi = u/r$ for the three different cases



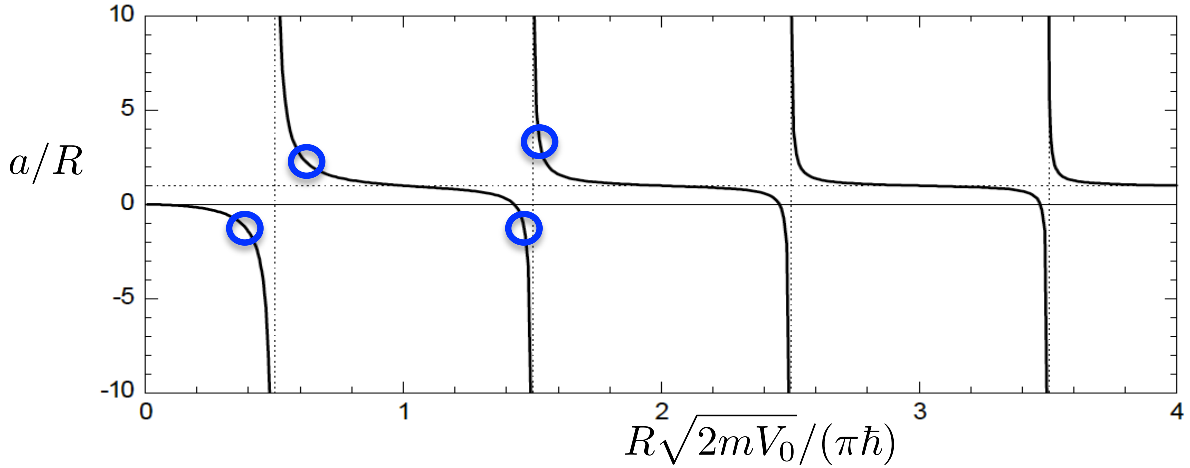
J. Walraven, Uni of Amsterdam (2009)

Summary ultracold scattering

- Only s-wave collisions contribute at low *T* (low *k*)
- The s-wave scattering length tells us about the phase shift of ψ (or *u*)



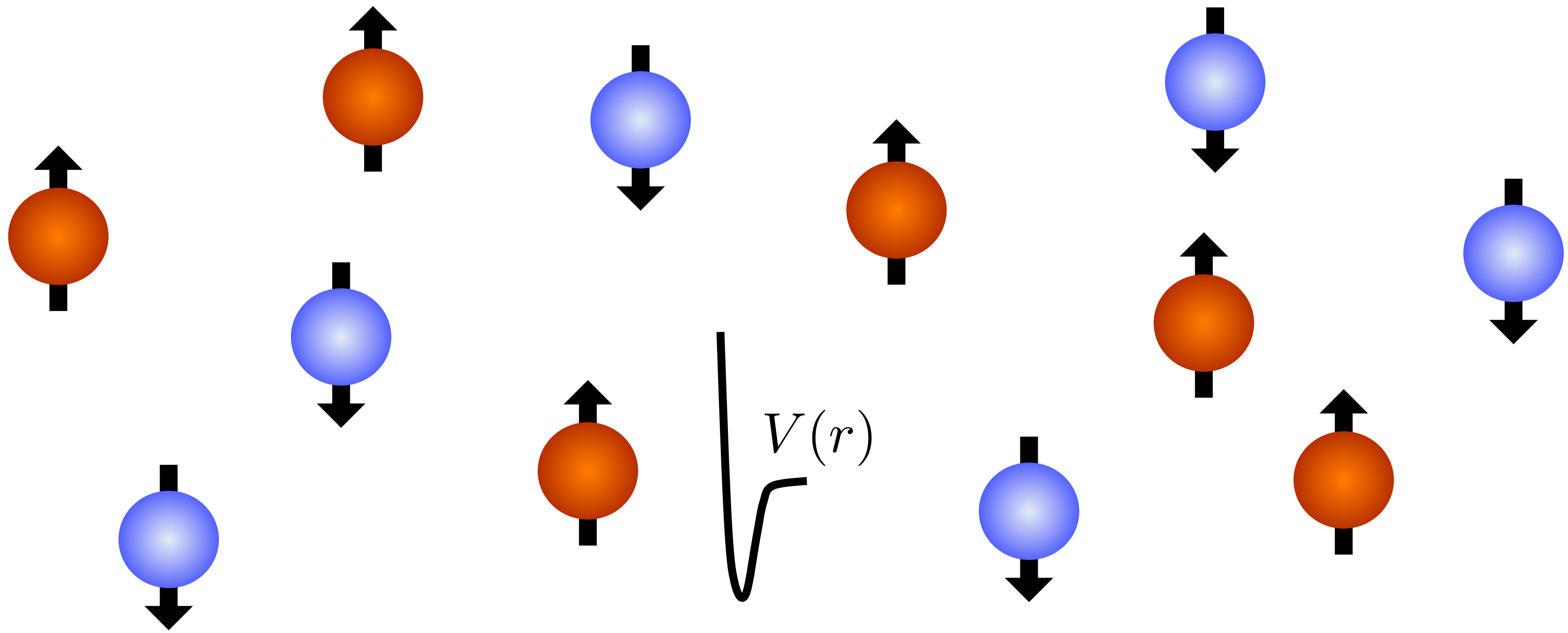
(M. Zweirlein, PhD, MIT, 2006)



- Positive *a* $\Rightarrow$ repulsive interaction, bound state near threshold
- Negative *a* $\Rightarrow$ attractive interaction, virtual state near threshold

Cold atom scattering

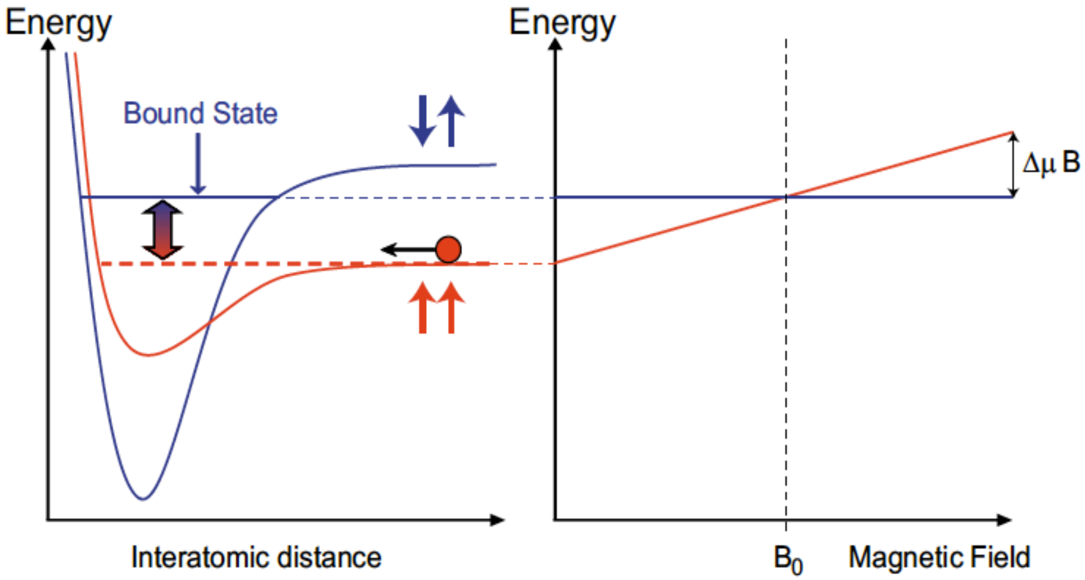
- Q: Can these phenomena occur for atoms interacting via a van der Waals potential ??



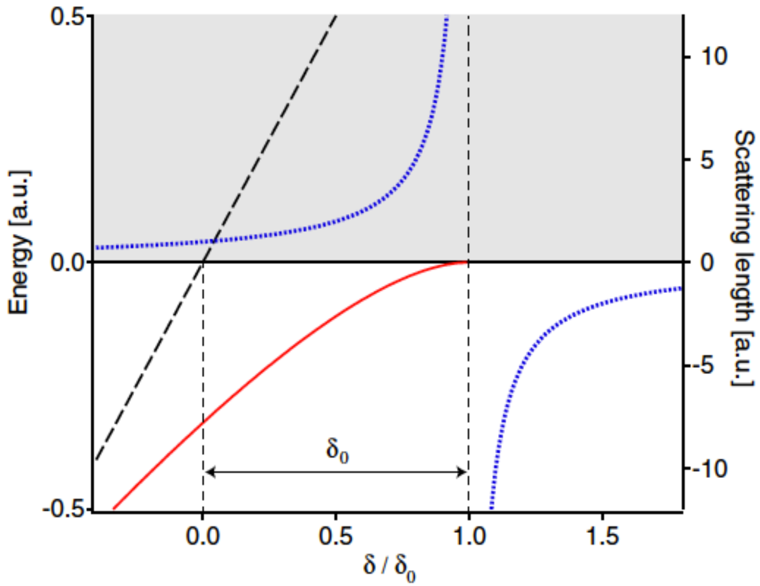
- A: Yes, using a Feshbach resonance…
- Note: λ_{dB} typically $\gtrsim n^{-1/3}$ and the range of $V(r)$ typically $\ll n^{-1/3}$
- This is a classic low energy scattering scenario where we don't resolve the details of $V(r)$

Feshbach resonances

- In a nutshell, hyperfine coupling mixes different combinations of internal atomic states (channels)
- These channels have different magnetic moments and can therefore be tuned relative to eachother by applying a magnetic field



(M. Zweirlein, PhD, MIT, 2006)



- Effectively equivalent to tuning the depth of a finite square well…
- Can tune scattering length through a pole and couple to a two-body bound state – introducing a bound state always leads to a pole in *a*

BEC-BCS crossover

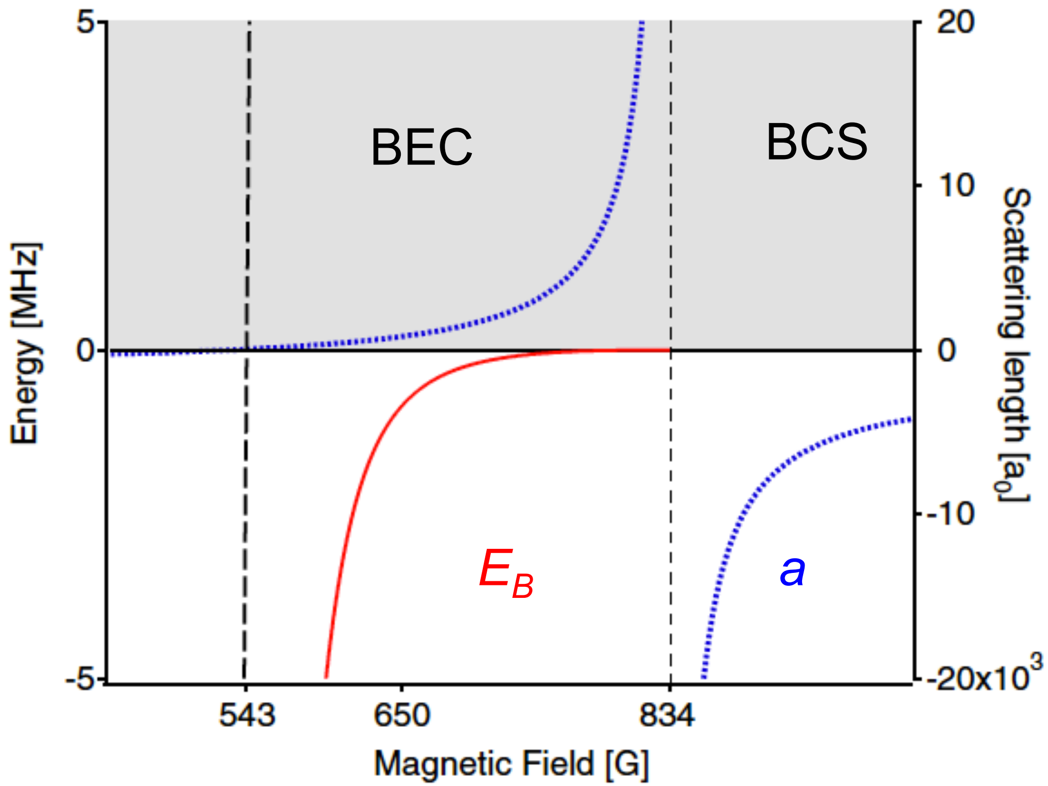
- The simplest way to get a feel for what's happening in the BEC-BCS crossover is to imagine a gas of spin-up and spin-down fermions that interact with each other via a finite square well

- We effectively have the ability to tune the depth of the potential starting from shallow (where the particles feel a weak attraction) to being deep enough to just support a bound state (where $a \rightarrow \infty$) to being even deeper such that the state is tightly bound and the two fermions act like a composite boson

- With this picture in mind we can now investigate what happens to a gas of atoms when the interactions are tuned in this way…

BEC-BCS crossover in Li-6

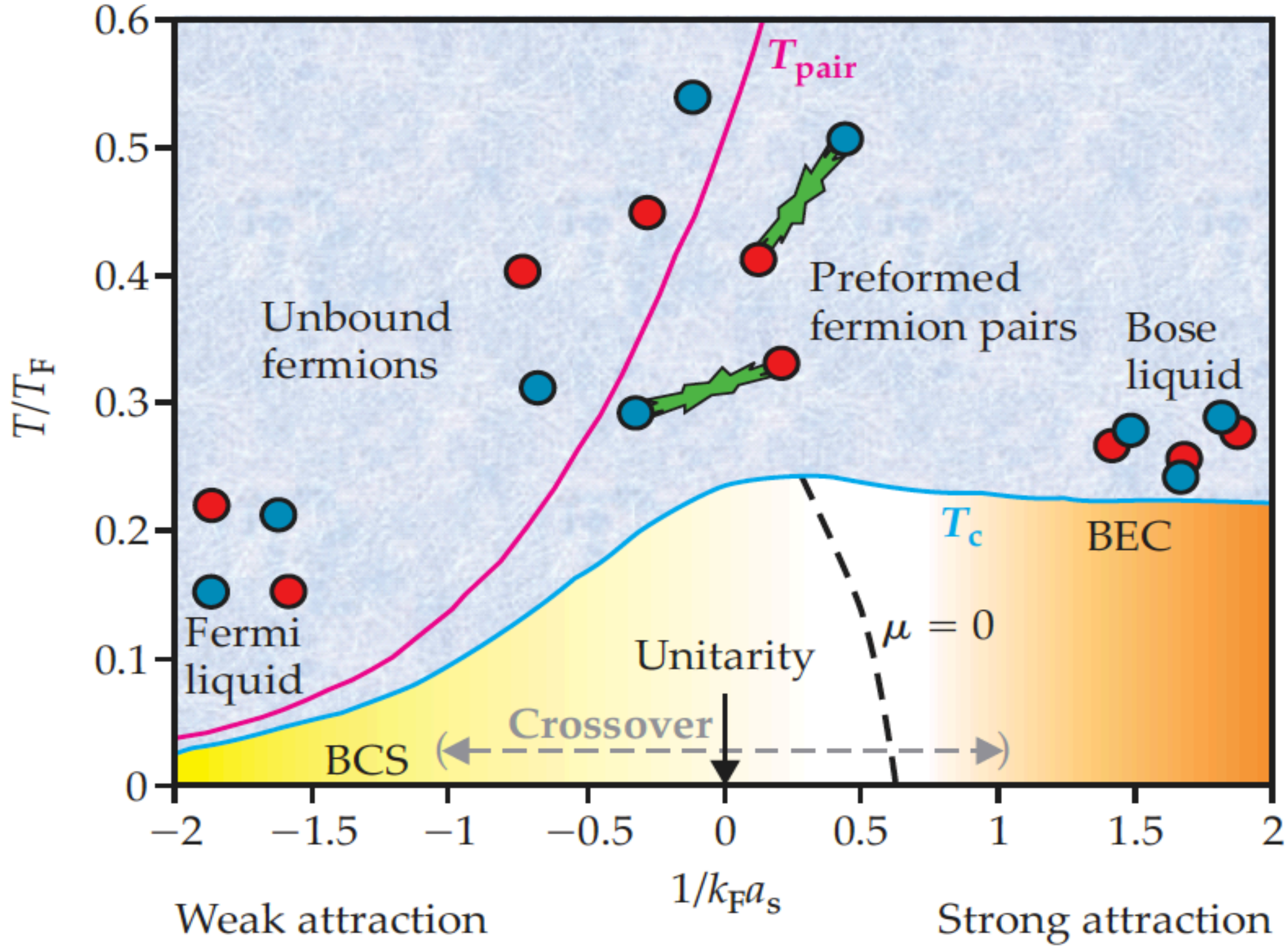
- We saw yesterday that turning up the depth of a finite square well leads to a bound state and pole in the scattering length
- In Li-6 (and also other atoms) the same features can be observed with a Feshbach resonance
- Li has an unusually broad resonance at 834 G which means the scattering length and hence E_B can be tuned with great precision
- An "ideal" realisation of the BEC-BCS crossover



(M. Zweirlein, PhD, MIT, 2006)

BEC-BCS crossover

- This image shows an approximate phase diagram of the crossover
- We can tune across it simply by varying *T* and the magnetic field



(Sa de Melo, Phys. Today, Oct 2008)

BEC vs BCS superfluidity…

- These two regimes can be visualised using dancers…

BEC of bound pairs

Loosely bound Cooper pairs

http://jila.colorado.edu/~jin/publications/images.html

BCS limit - Cooper Pairing

- We will now have a look at the properties of a gas in the different regions of the phase diagram

- The BCS regime is achieved when a is small and negative (weak attraction between spin-up/spin-down fermions) and there is no two-body bound state

- However, pairs can still form and condense into a superfluid (BCS) through a mechanism known as the Cooper instability

- We will now go through the physical arguments that lead to Cooper pairing and show how this gives a pairing gap

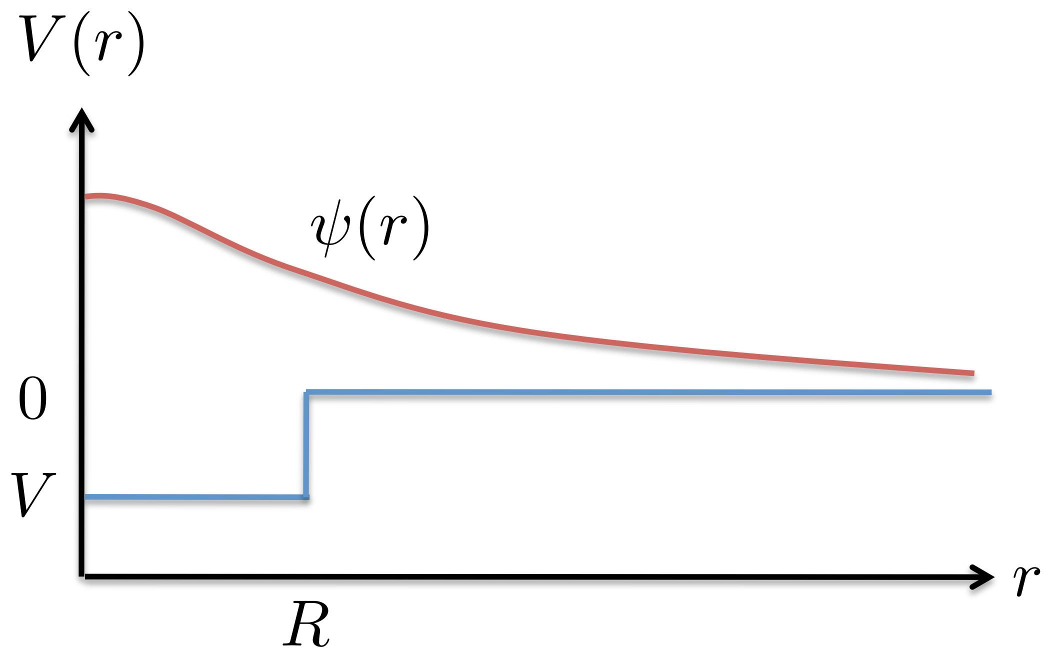
Bound states again…

- To understand Cooper pairing (and BCS superfluidity) we need to revisit the scattering problem and the conditions for achieving a bound state
- For reasons that will make sense later we'll look at the finite square well problem in 2D (as opposed to 3D)

- We will be interested in weakly bound states of size

$$r_B = 1/\kappa \equiv \sqrt{\frac{\hbar^2}{m|E_B|}} \gg R$$

- The potential must provide enough energy to change ψ' over a distance *R*



Two-body bound states – 2D

- In 2D the (radial) Schroedinger equation outside the well is:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) = \kappa^2\psi$$

- The solution is the modified Bessel which $\approx e^{-\kappa r}$ for large r
- When r is smaller ($R < r < 1/\kappa$) we can ignore κr which gives

$$\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) \approx 0$$

- This gives a logarithmic wavefunction which we normalise to 1 at R

$$\psi \approx \log(\kappa r)/\log(\kappa R)$$

- Inside the well ψ is ~ constant and $r\psi'$ changes from 0 to $1/\log(\kappa R)$
- The energy cost for this must be

$$\left(\frac{\Delta r\psi'}{\Delta R} = \frac{1}{\log(\kappa R)}\right) \qquad E \approx \frac{\hbar^2}{mR}\frac{\Delta r\psi'}{R} = \frac{E_R}{\log(\kappa R)} \qquad E_R = \frac{\hbar^2}{mR^2}$$

Two-body bound states – 2D

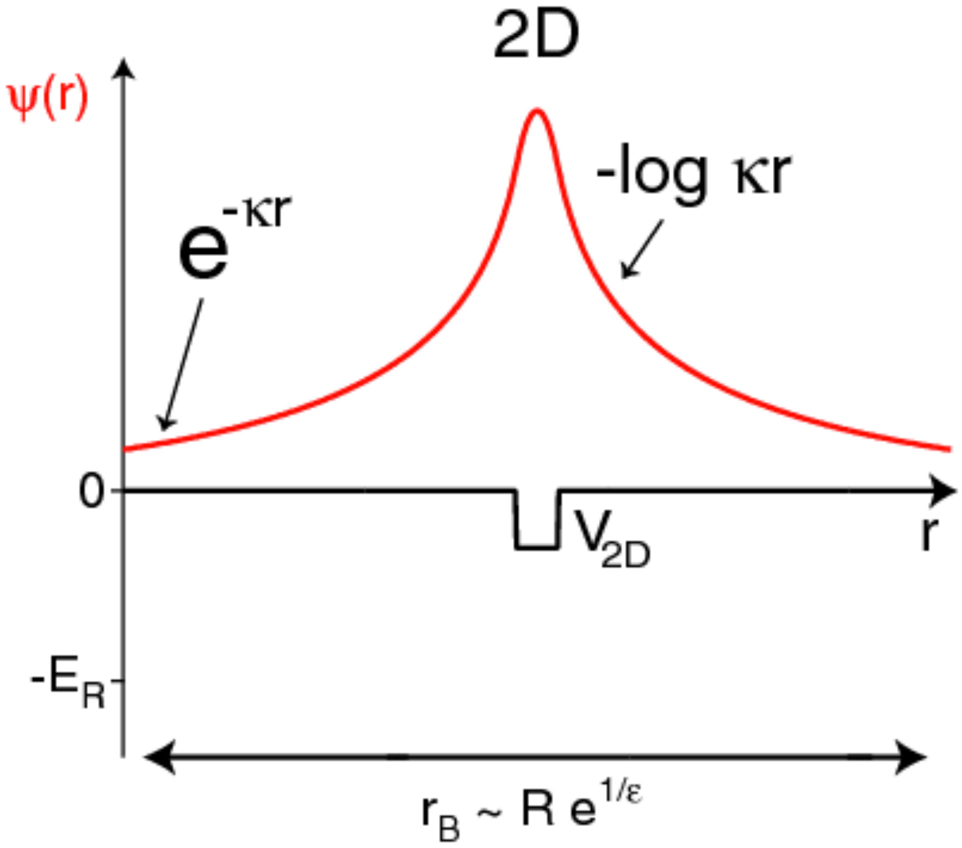
- Rearranging this to find κ: $$\kappa \approx \frac{1}{R} e^{-cE_R/V} \quad (c \sim 1)$$

- The binding energy is then

$$E_B \approx E_R e^{-cE_R/V}$$

- Note this is a very weakly bound state, depending exponentially on the small V
- The size of the bound state is exponentially large

- Unlike the 3D case, a bound state always exists for arbitrarily small V !!



(Ketterle & Zweirlein, Ultra-cold Fermi gases, Varenna 2007)

Two-body bound states – nD

- So in 2D (and 1D) there will always be a two-body bound state for arbitrarily weak interaction

- In 3D the attractive potential needs to exceed a critical depth before it can admit a bound state

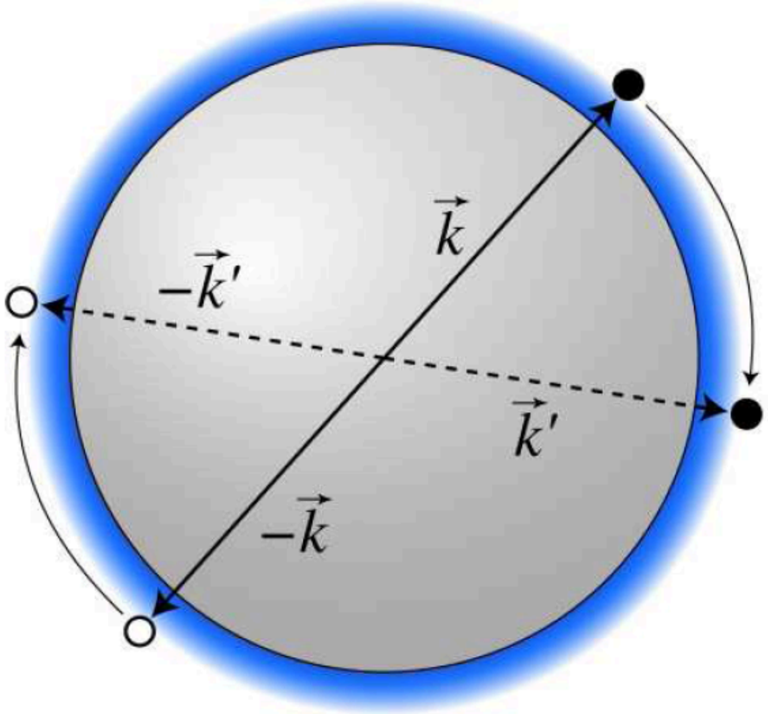
- How is it then that Cooper pairing which happens in 3D in the limit of very weak attractive interactions can happen??

- In 2D pairing can be understood at the two particle level which is clearly not the case in 3D however, and we therefore expect that pairing in 3D systems must somehow be a many-body effect

- Our next task is to identify what it is that leads to Cooper pairing (the bound states we've just studied will turn out to be a big help…)

Cooper Pairing

- In a low temperature ($T \rightarrow 0$) Fermi gas, elastic collisions are suppressed by Pauli exclusion as there are limited free states for particles to scatter into
- Consider two fermions with momenta on top of the Fermi sea
- The only states available for these particles to scatter into will be in a narrow shell near k_F



- In the small region near the Fermi surface the density of states is roughly constant, equal to

$$g_{3D}(E_F)$$

- We also get a constant density of states in 2D
- This has become a 2D problem…

(Ketterle & Zweirlein, Ultra-cold Fermi gases, Varenna 2007)

Cooper Pairing

- This simple picture gives an idea of why pairing might occur at the Fermi surface
- Imagine turning on the interaction from the top of the Fermi sea downwards, the lower energy particles will also form pairs
- The proper inclusion of all pairs in the many-body problem was achieved by Bardeen, Cooper and Schrieffer (1957) and is known as the BCS theory
- The end result is that the energy saving due to pairing is

$$\Delta = \frac{8}{e^2} E_F e^{-\pi/2k_F|a|}$$

- This is known as the pairing gap, it is the energy saved by forming pairs (note it is exponentially weak as we saw for 2D scattering)
- So we can understand Cooper pairing as a many-body effect which modifies the density of states to be constant like in 2D!!

Weakly Attractive Fermi gas ($a \rightarrow 0_-$)

- With weak attractive interactions, it is possible to calculate changes to the properties of the cloud perturbatively
- We will assume here that the local density approximation is valid

$$E = \int d\mathbf{r}\{\epsilon[n(\mathbf{r})] + V_{\mathrm{ho}}(\mathbf{r})n(\mathbf{r})\}$$

- The chemical potential also has a spatial dependence

$$\mu_0 = \mu[n(\mathbf{r})] + V_{\mathrm{ho}}(\mathbf{r}) \qquad \mu(n) = (3\pi^2)^{2/3}\frac{\hbar^2}{2m}n^{2/3}$$

- The density profile has the same shape as the ideal gas with a first order correction to the radius

$$R_i = \sqrt{\frac{2\mu_0}{m\omega_i^2}} = R_i^0\left(1 - \frac{256}{315\pi^2}k_F^0|a|\right)$$

Molecular BEC limit $(a \rightarrow 0_+)$

- On the opposite side of the Feshbach resonance, we produce tightly bound molecules that behave like point-like bosons
- This means they follow the GP equation etc etc
- The dimer-dimer (4-body) scattering problem is not obvious however, and was solved by Petrov et al., PRL (2004)

$$a_{\mathrm{dd}} = 0.60a$$

- When this scattering length is included in the GP equation the correct behaviour is recovered
- The bound molecules produced at a Feshbach resonance have a characteristic size of a much larger than r_0

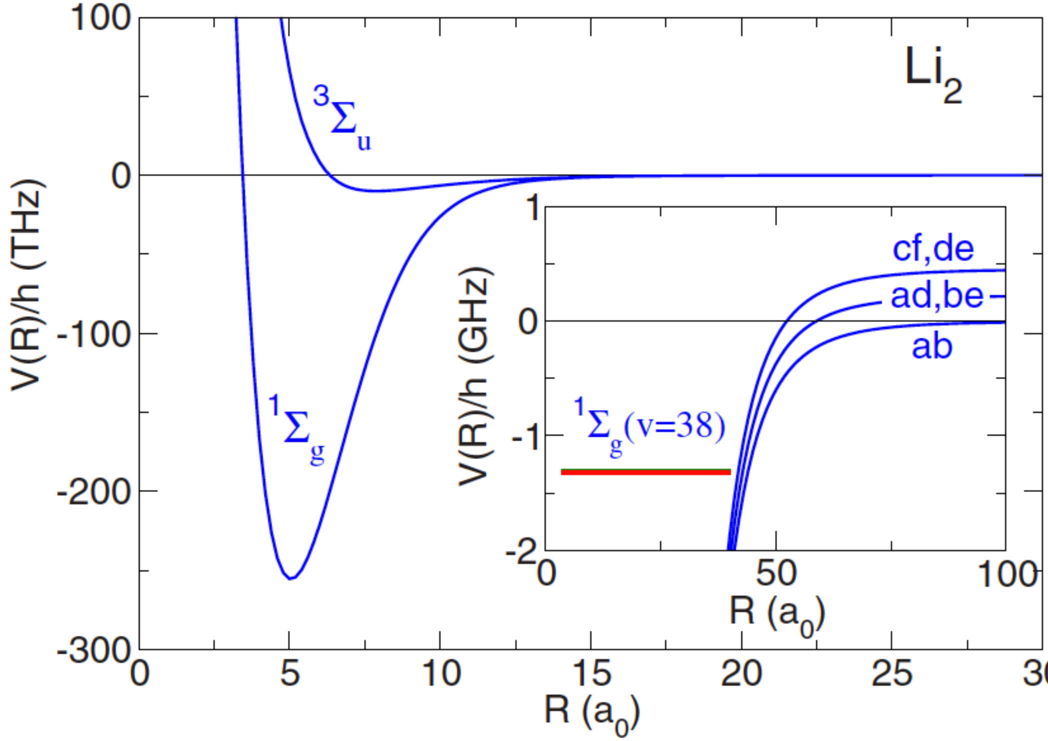
$$E_B = \frac{\hbar^2}{ma^2} \qquad \psi_b(r) = e^{-r/a}/\sqrt{2\pi a r}$$

- This is an important factor leading to their surprisingly long lifetimes

Weakly bound molecules

- One of the most remarkable features of fermion-fermion dimers formed near a Feshbach resonance (which came as a surprise) is their long lifetime (order 10 s) compared to boson pairs (1 ms)

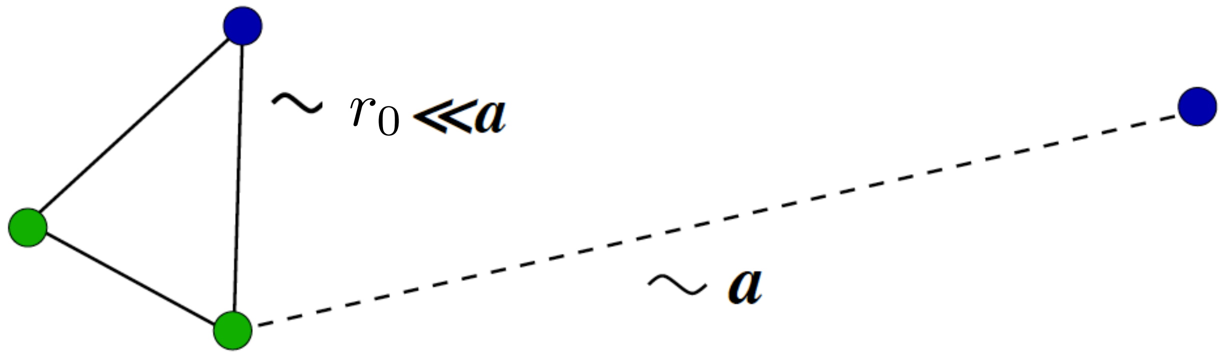
- These molecules are formed in the highest lying vibrational state and so should be highly unstable (any collision could lead to decay to the lower lying states)



C. Chin et al., Rev. Mod. Phys. 82, 1225 (2010)

Weakly bound molecules

- However, Pauli exclusion comes to the rescue..
- Molecular decay to a low lying vibrational level (size ~ r_0), requires three fermions coming together in a volume ~ r_0^3



- Two of these fermions are necessarily identical so this process will be Pauli suppressed by a factor involving r_0/a
- Petrov showed that the suppression factors are:

$$\alpha_{ad} \propto (r_0/a)^{3.33}$$

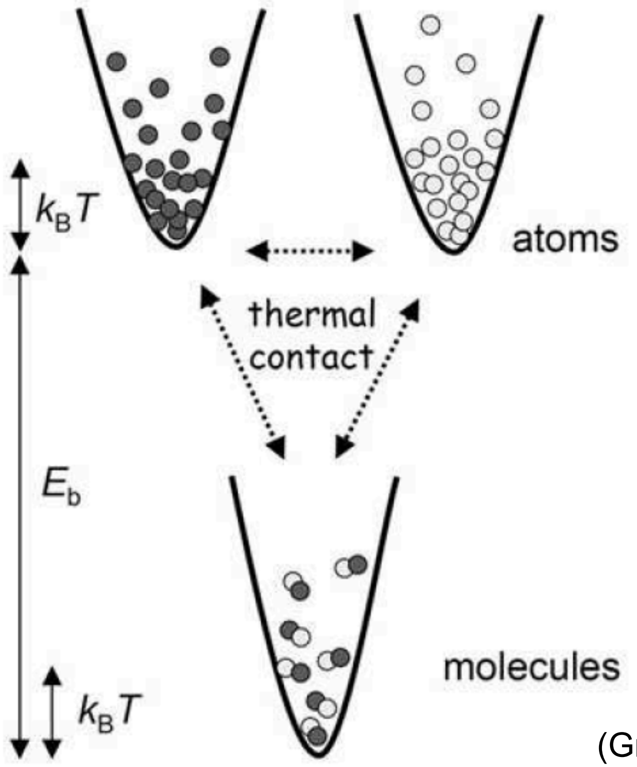
$$\alpha_{dd} \propto (r_0/a)^{2.55}$$

(Weaker binding = Longer lifetimes !?)

(Petrov et al., Ultra-cold Fermi gases, Varenna 2007)

Evaporation to MBEC

- These long lived molecules with small binding energy allow a neat way to produce a molecular BEC shown by the Innsbruck group by simply evaporating atoms (Chin, PRA 2004)
- How does this work ?? … Thermodynamics of course !!
- Consider the thermodynamic equilibrium of a mixture of trapped atoms and molecules with binding energy $E_b = \hbar^2/ma^2$



- Both molecules and atoms have the same harmonic confinement frequency
- Consider 3 types of particles spin_up/down atoms and molecules

$$N_\uparrow, N_\downarrow, M$$

- We also assume particle conservation

$$N_{Tot} = N_\uparrow + N_\downarrow + 2M$$

(Grimm, Ultra-cold Fermi gases, Varenna 2007)

Atom-Molecule equilibrium

- We solve this problem by writing down the total partition function (the product of each component)

$$Z = \sum_i e^{E_i/k_B T} \qquad\qquad Z_{Tot} = \prod_j Z_j$$

and then minimizing the Helmholtz Free Energy

$$F = -k_B T \log\left(Z_{Tot}\right)$$

- If we assume $N_\uparrow = N_\downarrow = N$ the total partition function can be written as below where we have divided it by the factorial of the number of atoms/molecules… (why ??)

$$Z_{Tot} = \frac{Z_N^{2N} Z_M^M \; e^{-ME_b/k_B T}}{(N!)^2 M!}$$

- Note that the molecular binding energy is also included for the *M* molecules

Atom-Molecule equilibrium

- We can then take the log of the partition function, invoke Stirling's rule for large *N*

$$\log(N!) = N\log(N) - N$$

- Eventually we'll get to an expression for Log(Z_{Tot}) that we can differentiate w.r.t. *N* giving

$$\frac{dF}{dN} = -k_B T(2\log(Z_N/N) - \log(Z_M/M) + E_b/k_B T)$$

- Setting this to zero (for the minimum) we can rearrange it to show

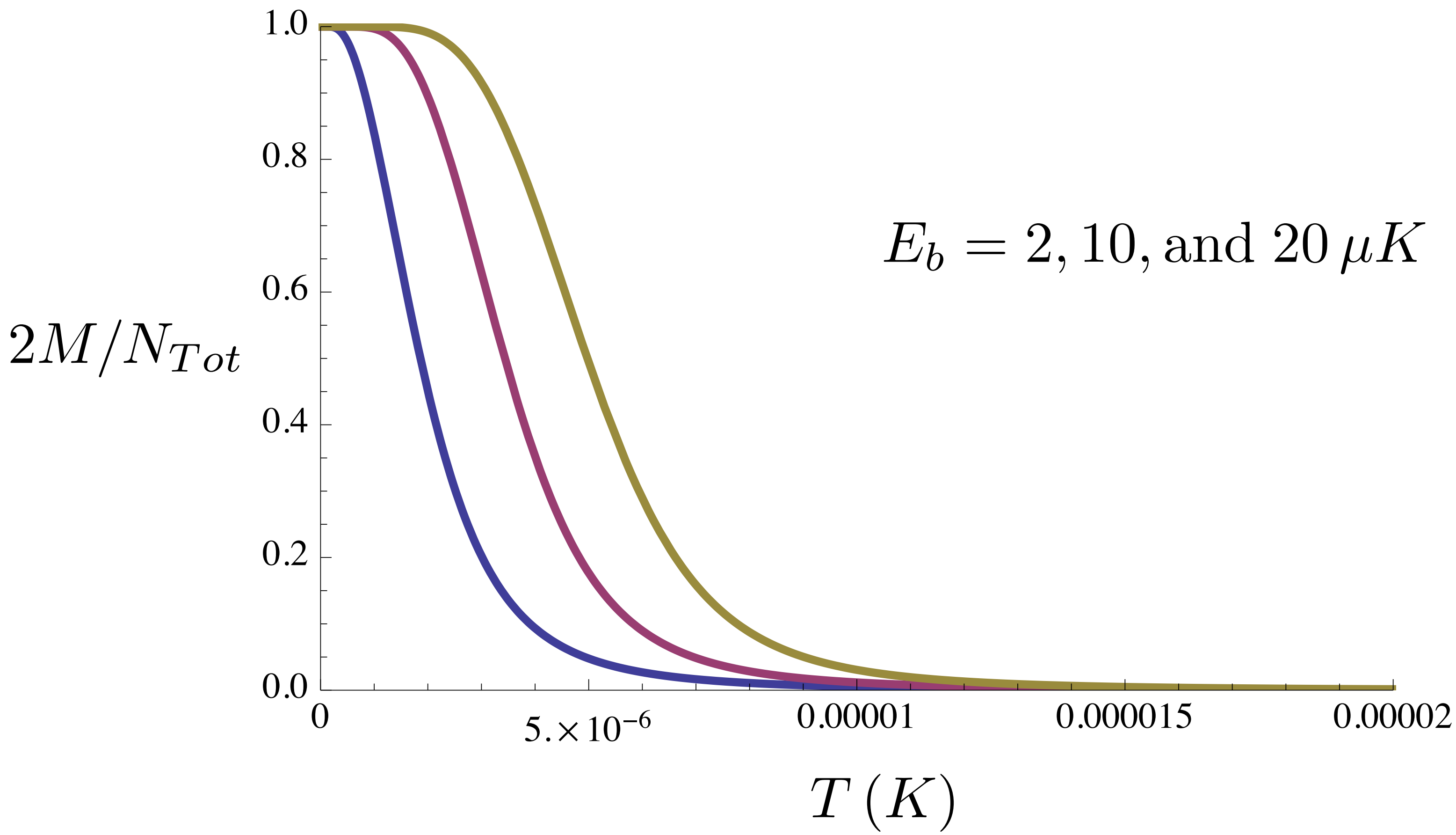
$$\phi_M = \phi_N^2 e^{E_b/k_B T}$$

where we have introduced the phase space density $\phi_i = N_i/Z_{N_i}$

- Note we have ignored Fermi degeneracy here so this analysis is really only valid for high *T* (> T_F)
- Nonetheless it shows that as we cool below E_b, the cloud will become molecular…

Atom-Molecule equilibrium

- Below is a plot showing typical behaviours of the molecule fraction as the temperature is lowered during evaporation



- The molecular fraction increases to unity at low *T* - for free !!

Unitarity Limit

- We have seen that the s-wave scattering length *a* diverges at a Feshbach resonance
- At this point it becomes much larger than any other relevant physical length scale in the problem ie. $a \gg n^{-1/3}, \lambda_{dB} \ldots$
- For this reason it no longer plays a role in the physics…
- The only parameters that determine the behaviours are the Fermi energy and the mean interparticle separation
- At *T* = 0 the chemical potential is scaled by a universal constant

$$\lim_{T \to 0} \mu_{Hom.} = \xi E_F \quad \text{(where)}\ \xi \approx 0.385$$

- Applying the local density approximation for the trapped gas gives

$$\lim_{T \to 0} \mu_{Tr.} = \sqrt{\xi} E_F$$

- Other properties of the gas are scaled universally in a similar way

Summary of all regimes

- The key properties of all three regimes are summarised below
- I'm not going to dwell on how each of these are derived as you can look them up and read about them in a variety of references

$\frac{1}{k_F a}$	BEC-limit ∞	Unitarity 0	BCS-limit $-\infty$
γ (in $\mu \propto n^\gamma$)	1	$2/3$	$2/3$
$n_\uparrow(\mathbf{r})/n_\uparrow(\mathbf{0})$	$1 - \sum_i \frac{x_i^2}{R_i^2}$	$(1 - \sum_i \frac{x_i^2}{R_{Ui}^2})^{3/2}$	$(1 - \sum_i \frac{x_i^2}{R_{Fi}^2})^{3/2}$
$n_\uparrow(\mathbf{0})$	$\frac{15}{8\pi} \frac{N_\uparrow}{R_x R_y R_z}$	$\frac{8}{\pi^2} \frac{N_\uparrow}{R_{Ux} R_{Uy} R_{Uz}}$	$\frac{8}{\pi^2} \frac{N_\uparrow}{R_{Fx} R_{Fy} R_{Fz}}$
Radii	$R_i = \sqrt{\frac{2\mu M}{M\omega_i^2}}$	$R_{Ui} = \xi^{1/4} R_{Fi}$	$R_{Fi} = \sqrt{\frac{2E_F}{m\omega_i^2}}$

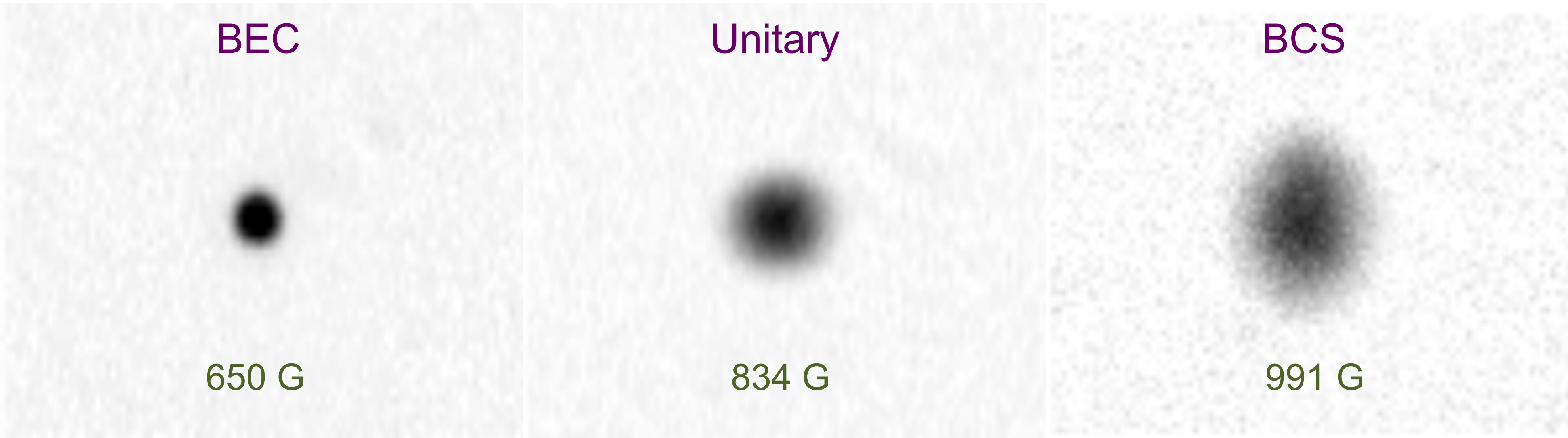
$(\xi = 1 + \beta)$

(Ketterle & Zweirlein, Ultra-cold Fermi gases, Varenna 2007)

- These three regimes are smoothly connected as we've discussed theoretically
- What does this mean practically ???

Structure of an interacting Fermi gas

- We now have a simple picture of a two component Fermi gas interacting via *s*-wave scattering in the BEC-BCS crossover



$$N_\sigma = 1.5 \times 10^5 \qquad \frac{T}{T_F} \lesssim 0.1 \;\; \text{(at Unitarity)}$$

- Absorption images show the change in density profile (cloud gets bigger and more energetic on BCS side of resonance)
- No obvious way to distinguish pairs from free atoms which is where our research comes into play…

Universal behaviours

- It has also been shown that when the scattering length is large the physical properties of the gas can become universal
- A number of exact relations have been derived for gases in this universal regime which are surprisingly simple in that they are analytic results that link microscopic and macroscopic quantities
- These have become known as the Tan relations after their inventor
- Some examples are…

$$\mathcal{P} = \frac{2}{3}\mathcal{E} + \frac{\hbar^2}{12\pi m a}\, C$$

$$\left(\frac{\mathrm{d}E}{\mathrm{d}a^{-1}}\right)_S = -\frac{\hbar^2}{4\pi m} C$$

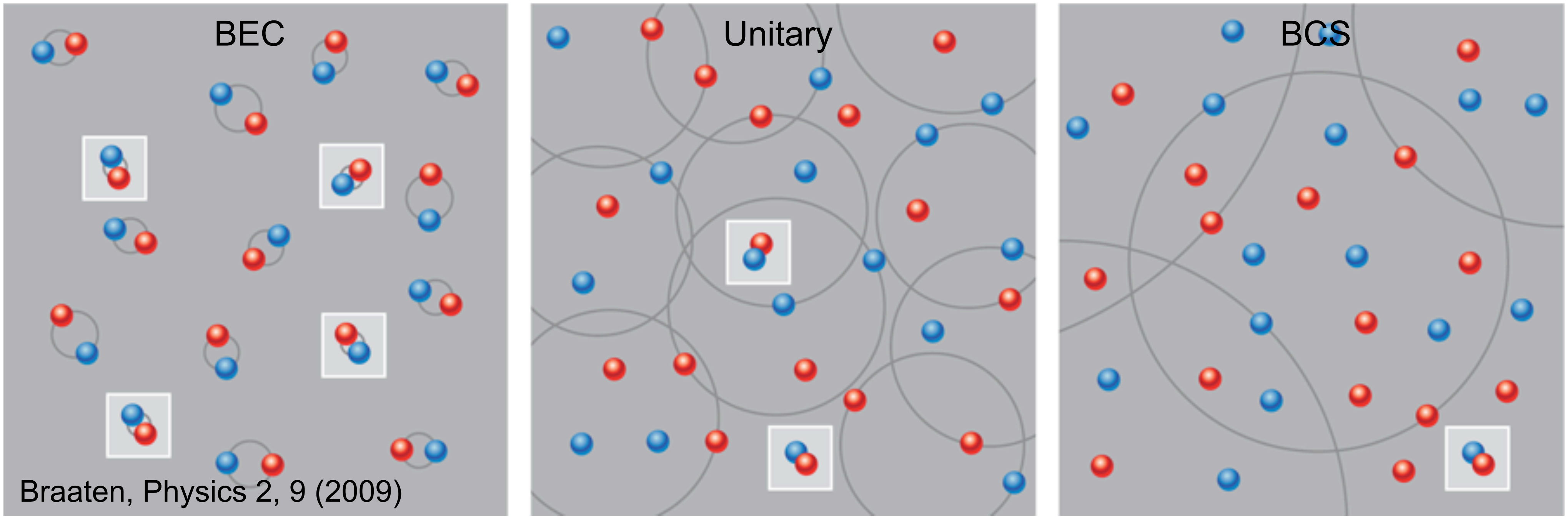
$$T + U - V = -\frac{\hbar^2}{8\pi m a}\, C$$

$$\left\langle n_1\left(\boldsymbol{R} + \frac{1}{2}\boldsymbol{r}\right) n_2\left(\boldsymbol{R} - \frac{1}{2}\boldsymbol{r}\right)\right\rangle \longrightarrow \frac{1}{16\pi^2}\left(\frac{1}{r^2} - \frac{2}{ar}\right)\mathcal{C}(\boldsymbol{R})$$

Tan, (2005)

Universal behaviours

- Note that all of these relations involve the contact parameter C
- This single parameter contains all of the many-body physics…
- So what is it ??



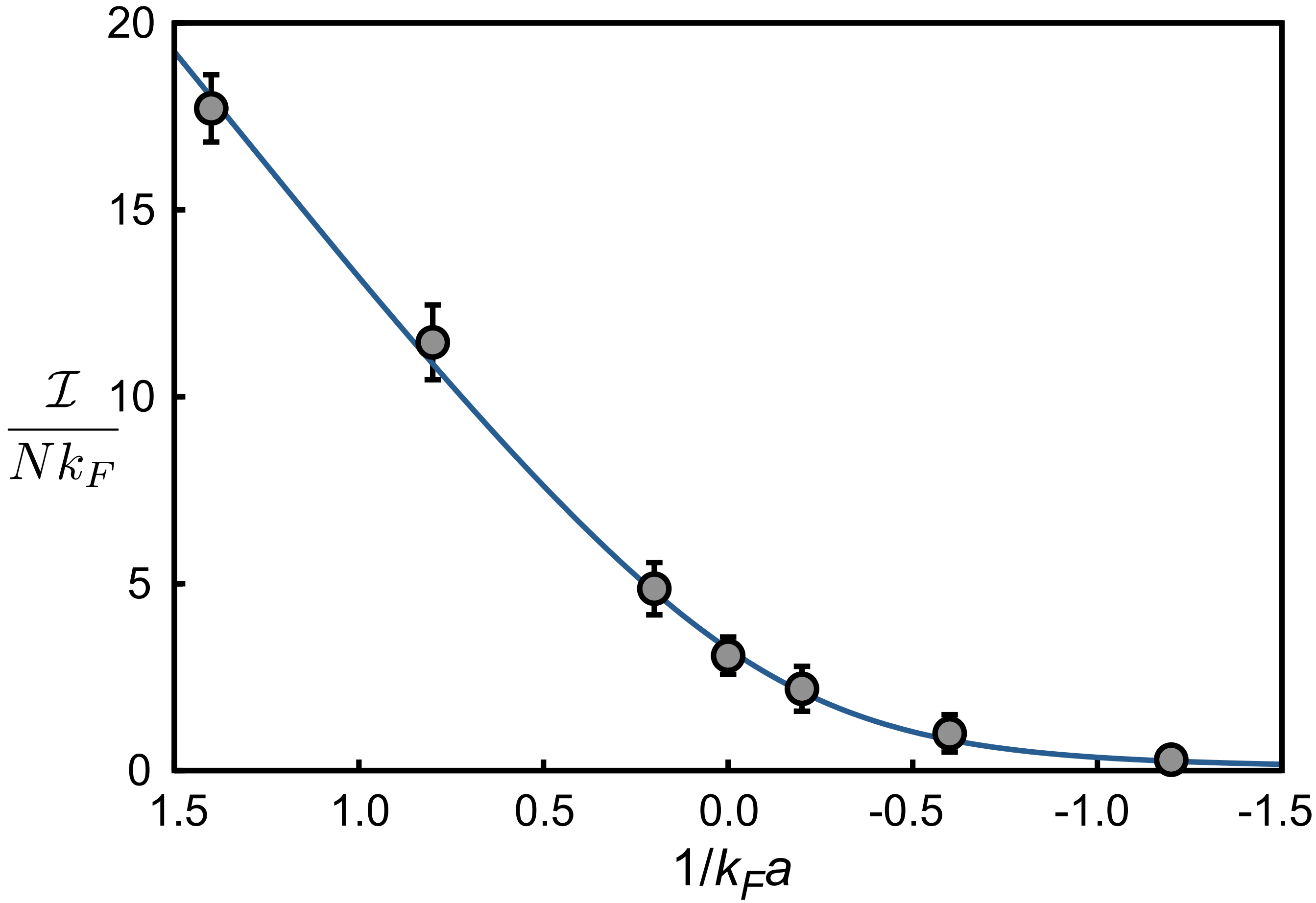
- C quantifies the number of closely spaced pairs

$$\mathcal{C} = \lim_{k\to\infty} k^4 n(k)$$

Universal contact

- Our expression for the structure factor can easily be rearranged to give the contact as a function of $S(k)$

$$\frac{\mathcal{I}}{Nk_F} = \left(\frac{k}{k_F}\right) \frac{4S_{\uparrow\downarrow}(k)}{1 - \frac{4}{\pi k a}}$$



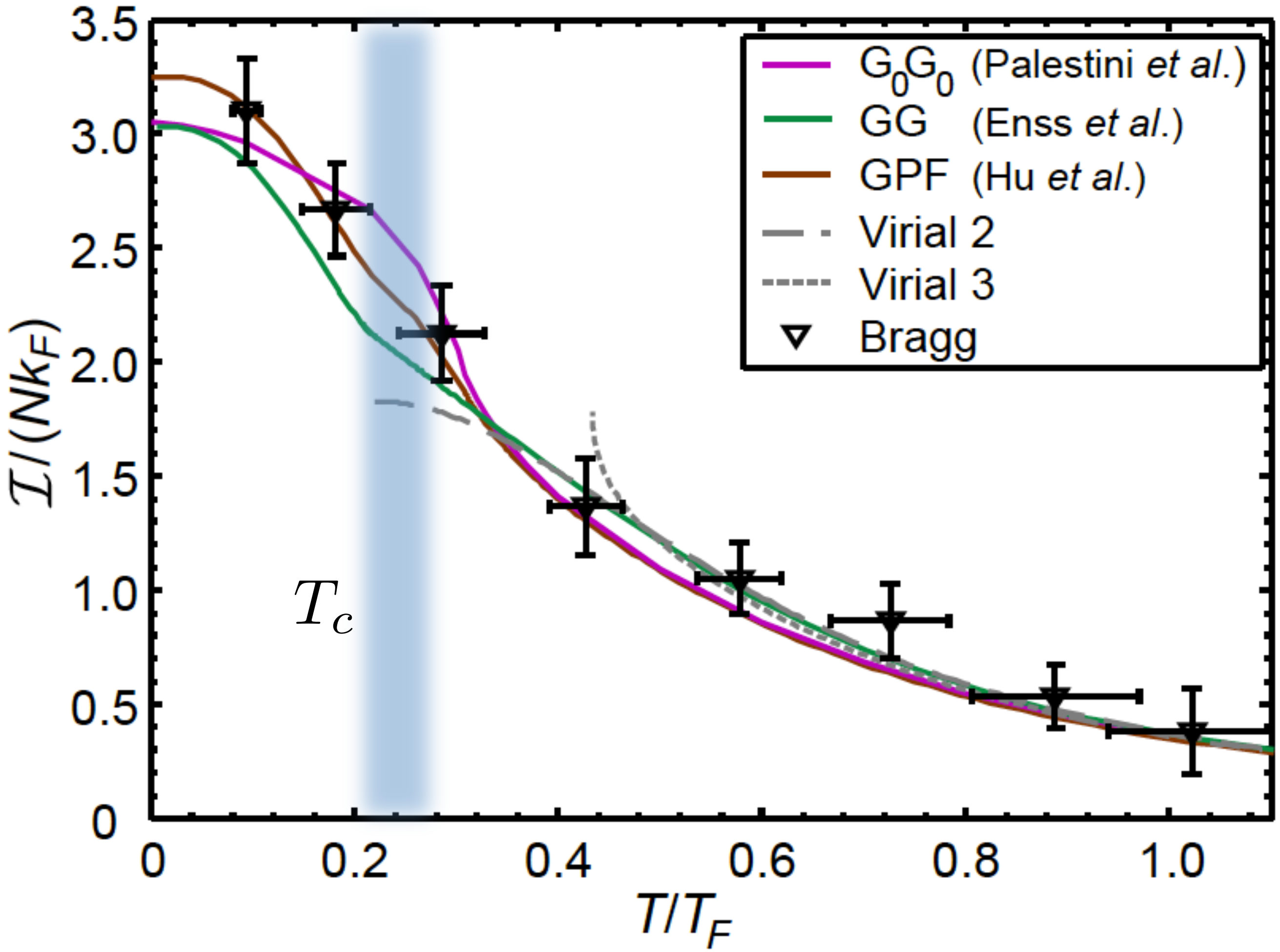
- This expression appears to hold over a surprisingly broad region of the Feshbach resonance…

Kuhnle *et al.*, (2010)

Universal contact – *T* dependence

- At unitarity $\frac{1}{k_F a} = 0$ so the contact simplifies to

$$\frac{\mathcal{I}}{Nk_F} = \left(\frac{k}{k_F}\right) 4S_{\uparrow\downarrow}(k)$$



- This is the first measure of the temperature dependence of the contact in a unitary gas
- Shows quantitatively the build up of pair correlations

Kuhnle *et al.*, Phys Rev Lett (2011)