Physics of ultracold Bose gases in one-dimensional and ring traps

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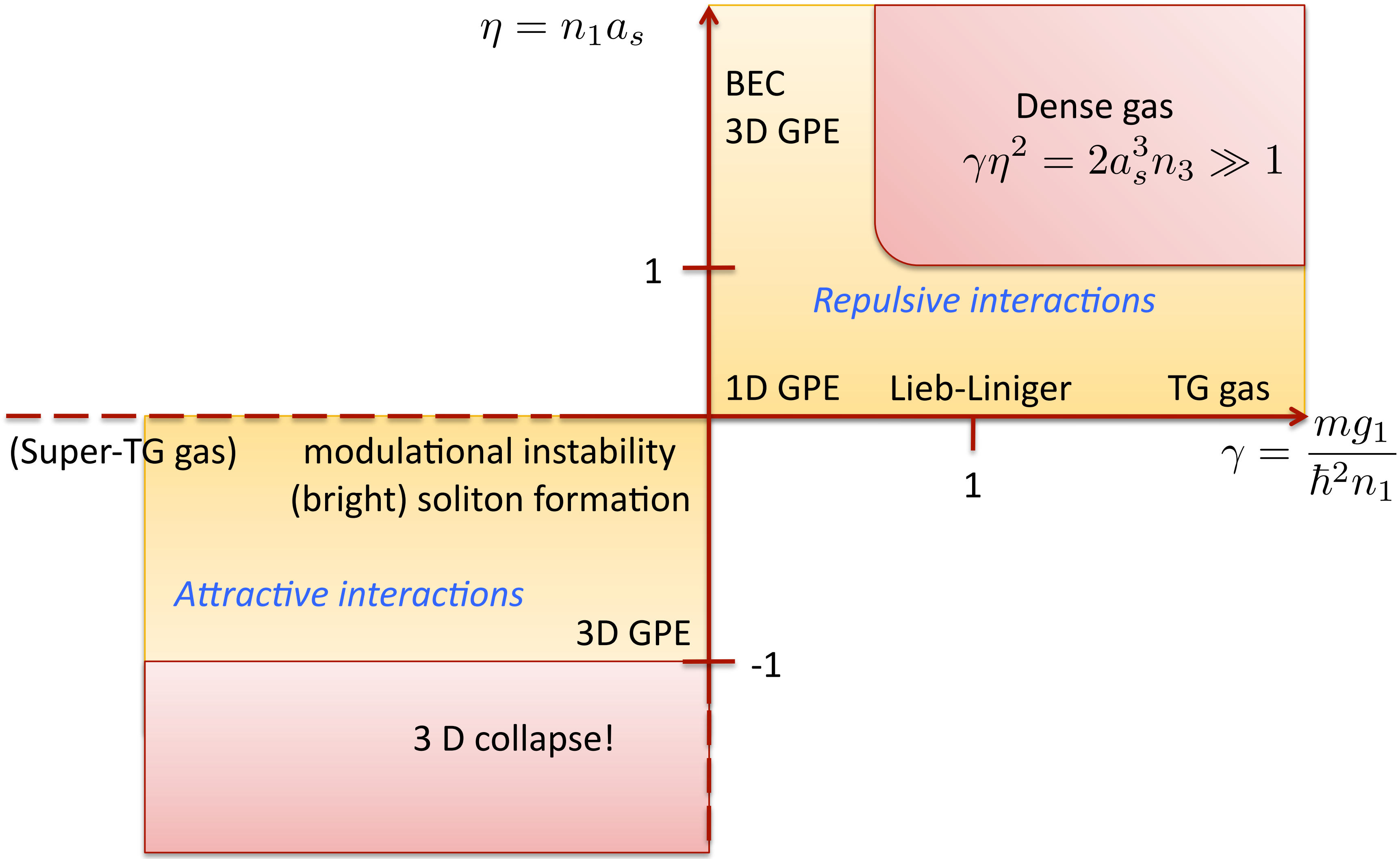
Lecture II - Victorian Summer School in Ultracold Physics (VSSUP) 2012

One dimension is different!

- To be covered in these lectures:
  - Absence of true Bose-Einstein condensation
  - Strongly-correlated many-body physics with a dilute gas
  - Bosons play fermions
  - Superfluid or not superfluid (or maybe both?)
  - Schrödinger cats made robust
  - [Stirring up solitons]

Interaction strength and dimensionality



Expect 1D physics when $|\eta| \ll 1$

The 1D gas can be dilute even when $\gamma \gg 1$ -> strong correlation

Phase fluctuating condensate?

- Bogoliubov's trick:

$$\hat{\psi}(x) = \phi(x) + \delta\hat{\psi}(x)$$

This obviously works if BEC is present (3D).

However, it is sufficient to have small density fluctuations (works in 1D without BEC):

$$\hat{\rho}(x) = \hat{\psi}^\dagger(x)\hat{\psi}(x) \approx \rho_0 + \delta\hat{\rho}(x)$$

The (fluctuating) phase is then "defined" by

$$\hat{\psi}(x) = \sqrt{\hat{\rho}}e^{\hat{\theta}}$$

Y. Castin, **Simple theoretical tools for low dimension Bose gases**, J. Phys. IV France, 116, 89 (2004) arXiv:0407118

V. N. Popov, **Functional Integrals in Quantum Field Theory and Statistical Physics**, (Reidel, Dordrecht, 1983).

Bosons play fermions

The Lieb-Liniger model and the Tonks-Girardeau gas

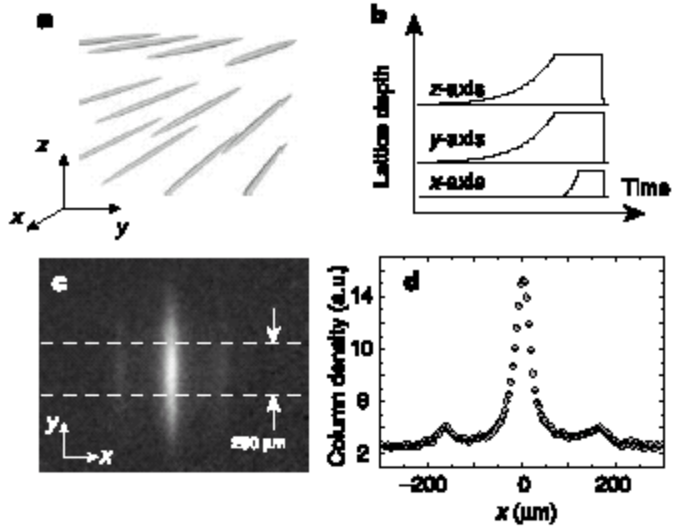
Tonks-gas – Experiments

letters to nature

Tonks–Girardeau gas of ultracold atoms in an optical lattice

Belén Paredes[1], Artur Widera[1,2,3], Valentin Murg[1], Olaf Mandel[1,2,3], Simon Fölling[1,2,3], Ignacio Cirac[1], Gora V. Shlyapnikov[4], Theodor W. Hänsch[1,2] & Immanuel Bloch[1,2,3]

MPQ Garching



up to γ_{eff}~ 200

other experiments:

T. Esslinger (Zürich)

W. Phillips (NIST)

D. Weiss (PSU), γ~5.5

R. Grimm (Innsbruck): confinement induced resonance!

$$\gamma \approx \frac{\text{interaction energy}}{\text{kinetic energy}}$$

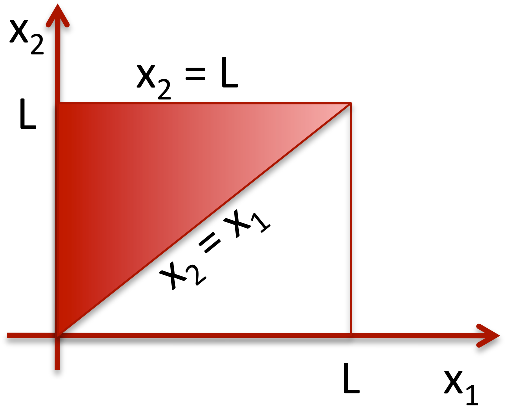
$$\gamma \simeq \frac{m}{\hbar}\frac{\omega_\rho}{n_{1D}} a_{3D}$$

1D Bose Gas – Lieb-Liniger model

$$H = \sum_i \frac{p_i^2}{2m} + 2g_{1\mathrm{D}} \sum_{i<j} \delta(x_i - x_j)$$

- 1D Bosons with repulsive δ interactions
- Ground- and excited-state wavefunctions exactly known from Bethe ansatz [Lieb, Liniger (1963)]
- Interaction parameter $\gamma = \frac{m}{\hbar^2} \frac{g_{1\mathrm{D}}}{n}$
- For $\gamma \to \infty$, problem is mapped exactly to free Fermi gas (Tonks-Girardeau gas) [Girardeau (1960)]
- Ring geometry provides periodic boundary conditions

Lieb-Liniger model: wave function



Consider $0 \leq x_1 \leq x_2 \cdots \leq x_N \leq L$

- Inside: $$-\frac{\hbar^2}{2m}\sum_i \frac{\partial^2}{\partial x_i^2}\psi = E\psi$$
- Boundary conditions are provided by
  - Interactions
  - Periodicity in the box
- Bethe ansatz:

$$\psi(x_1,\ldots,x_N) = \sum_P a(P) P \exp(i\sum_{j=1}^{N} k_j x_j)$$

P is a permutation of the set $\{k\} = k_1, k_2, \ldots, k_N$

- Just one quasimomentum per particle (!)
- Model is integrable, check Yang-Baxter equation

Bose-Fermi mapping

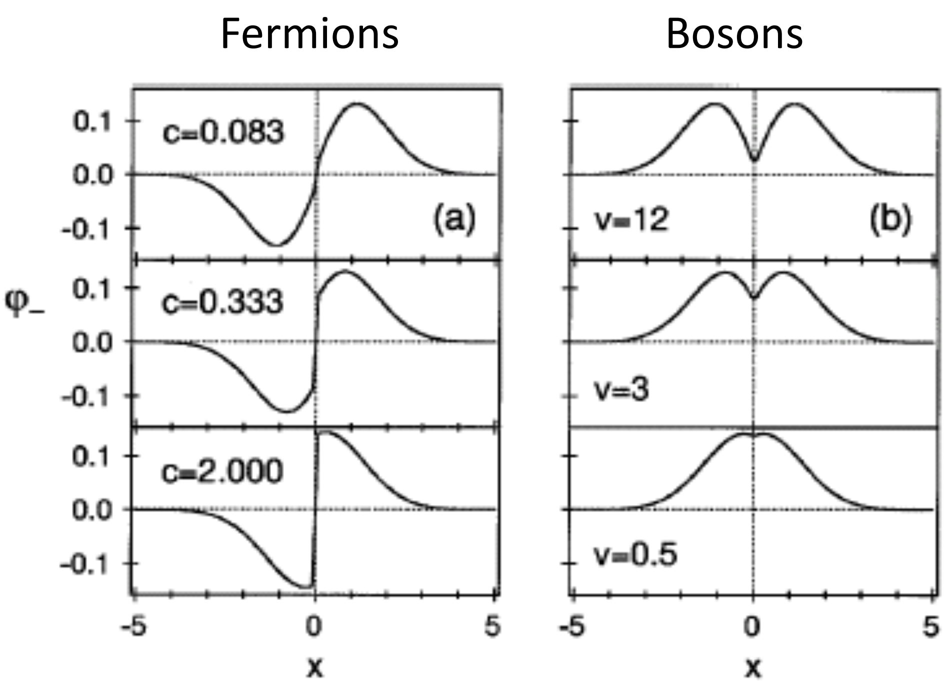
"In 1D, there is no distinction between Bosons and Fermions"

Strong repulsive interactions for bosons have the same effect as the Pauli exlusion principle for fermions.

The 1D Bose gas maps one-to-one to a gas of spinless fermions

$$\phi^B = |\phi^F|$$

Bosons with strong but finite interactions map to spinless fermions with weak short-range interactions



Cheon and Shigehara, 1999

Girardeau, 1960

Pseudopotential in the Fermionic picture

Sen's pseudopotential generates correct energy levels to first order in $1/\gamma$

$$V(x_1, x_2) = -\frac{2\hbar^2}{mc}\delta''(x_1 - x_2)$$ [D. Sen 1999]

generalization for arbitrary γ:

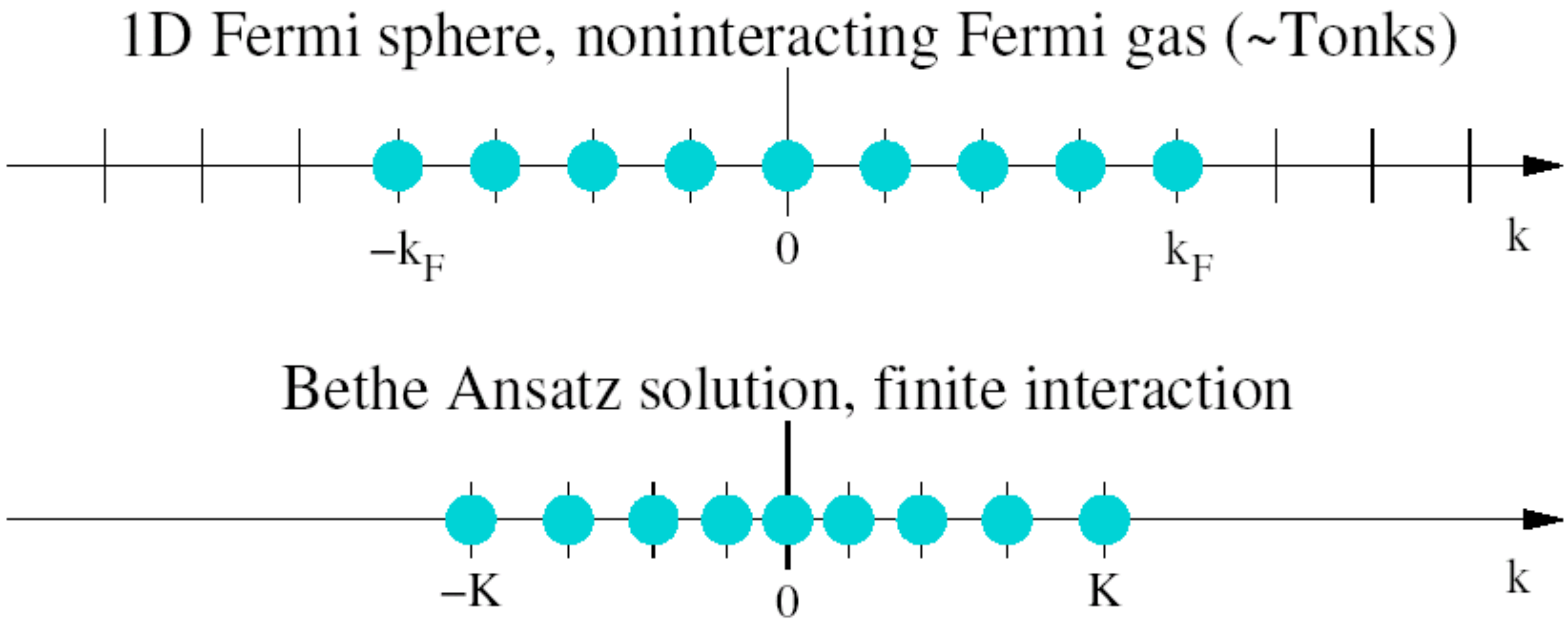
$$V(x_1, x_2, x_2', x_1') = -\frac{4\hbar^2}{mc}\delta\left(\frac{x_1 + x_2 - x_2' - x_1'}{2}\right)\delta'(x_1 - x_2)\delta'(x_1' - x_2')$$

Granger and Blume [2004],
Girardeau and Olshanii [2004],
Brand and Cherny [2005]

This can be used to apply common methods of fermionic many-body theory, e.g.

- Hartree-Fock
- diagrammatic many-body perturbation theory
- Random-phase approximation

The nature of Bethe-ansatz solutions: Quasi-momenta and Fermi sphere



Total energy: $$E = \frac{\hbar^2}{2m} \sum_k k^2$$

Total momentum: $$P = \hbar \sum_k k$$

Lieb-Liniger ground states

The quasi momentum distribution in the ground state is deformed from the simple Fermi-sphere picture at finite (weaker) interactions

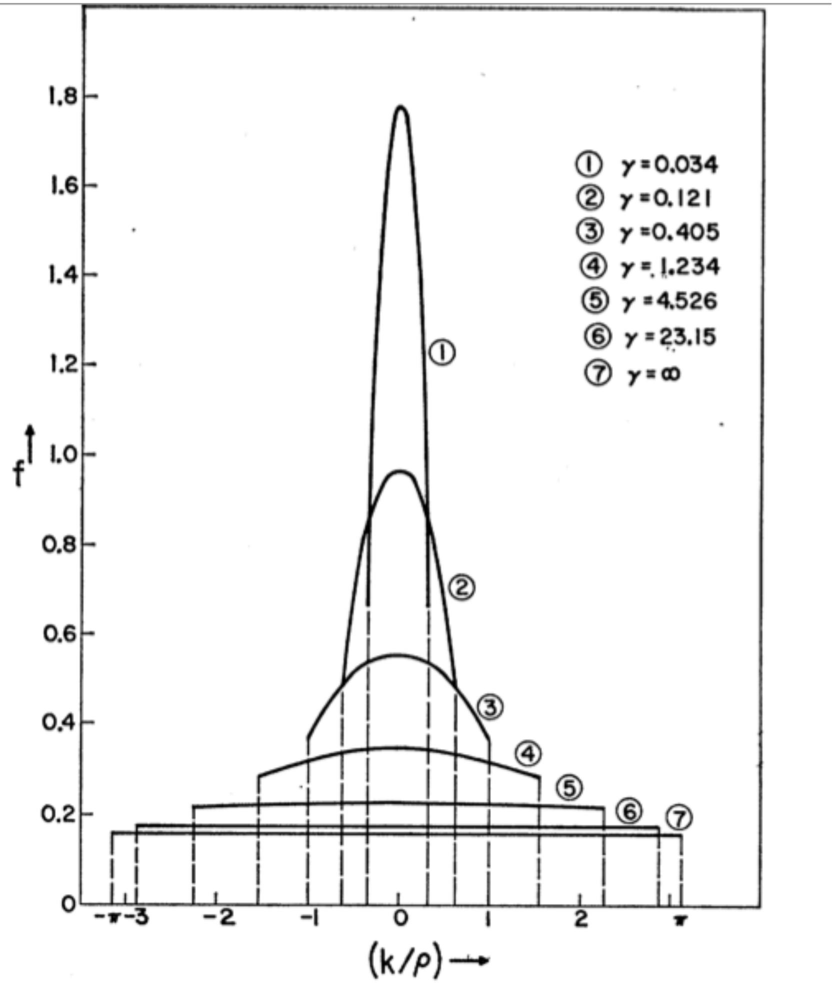


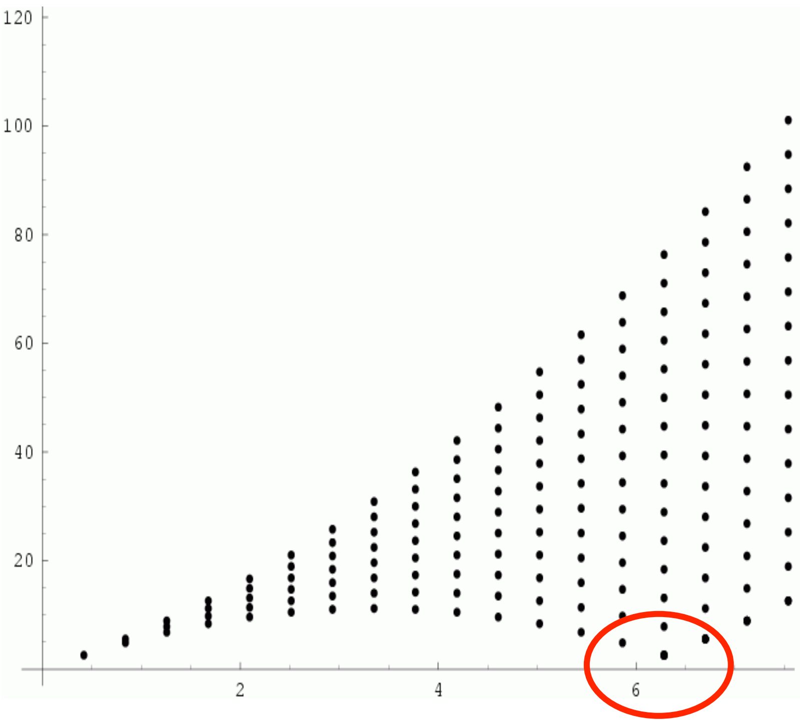
FIG. 2. The distribution function of "quasi-momenta" in the ground state for various values of $\gamma = c/\rho$. The vertical dashed lines are the cutoff momenta K (cf. Fig. 1). When $\gamma = \infty$, $f = (2\pi)^{-1}$. For all γ, $\int_{-K}^{K} f(k)dk = \rho$.

Lieb, Liniger, 1963

Excitation spectrum for the Lieb-Liniger model

$\gamma = \infty$

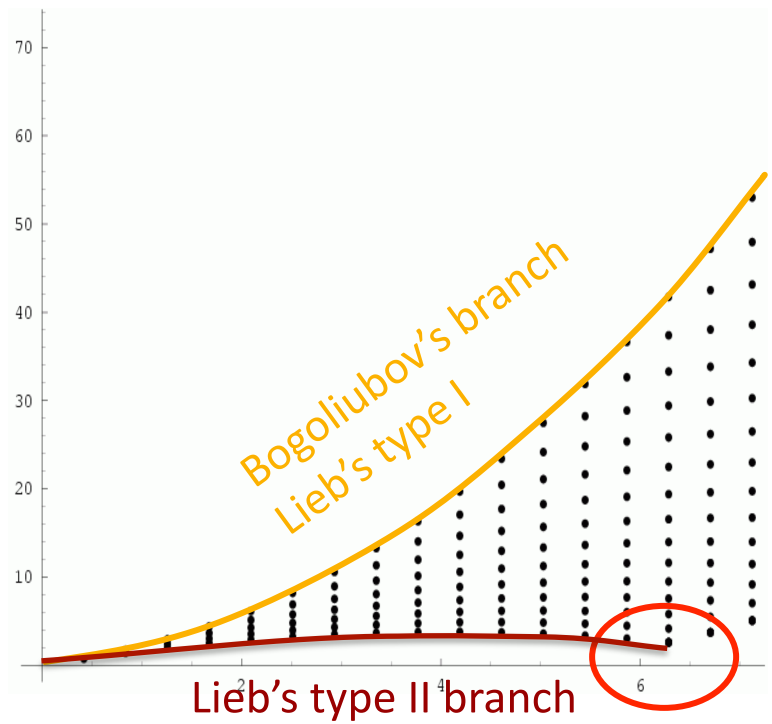
$\sim\omega\pi^2/\varepsilon_F$



momentum $q\pi/k_F$

$\gamma=1$

$\sim\omega\pi^2/\varepsilon_F$



momentum $q\pi/k_F$

umklapp excitation

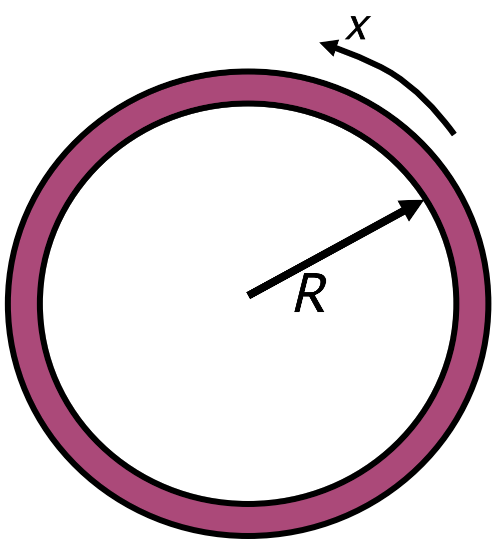
$k_F=\pi n_{1D}$; $\varepsilon_F=\hbar^2 k_F^2/(2m)$

Type II excitations can be identified with dark solitons!

Superfluid or not?

Or maybe both?

1D Bose gas on a ring



Circumference: $L = 2\pi R$

We work with the Lieb-Liniger model, i.e. Bosons with contact interactions in one dimension

What are the superfluid properties of the Bose gas on the ring at zero temperature?

- non-classical rotational inertia or Hess-Fairbank effect:

  Does the fluid in equilibrium take part in small (infinitesimal) rotations of the container?

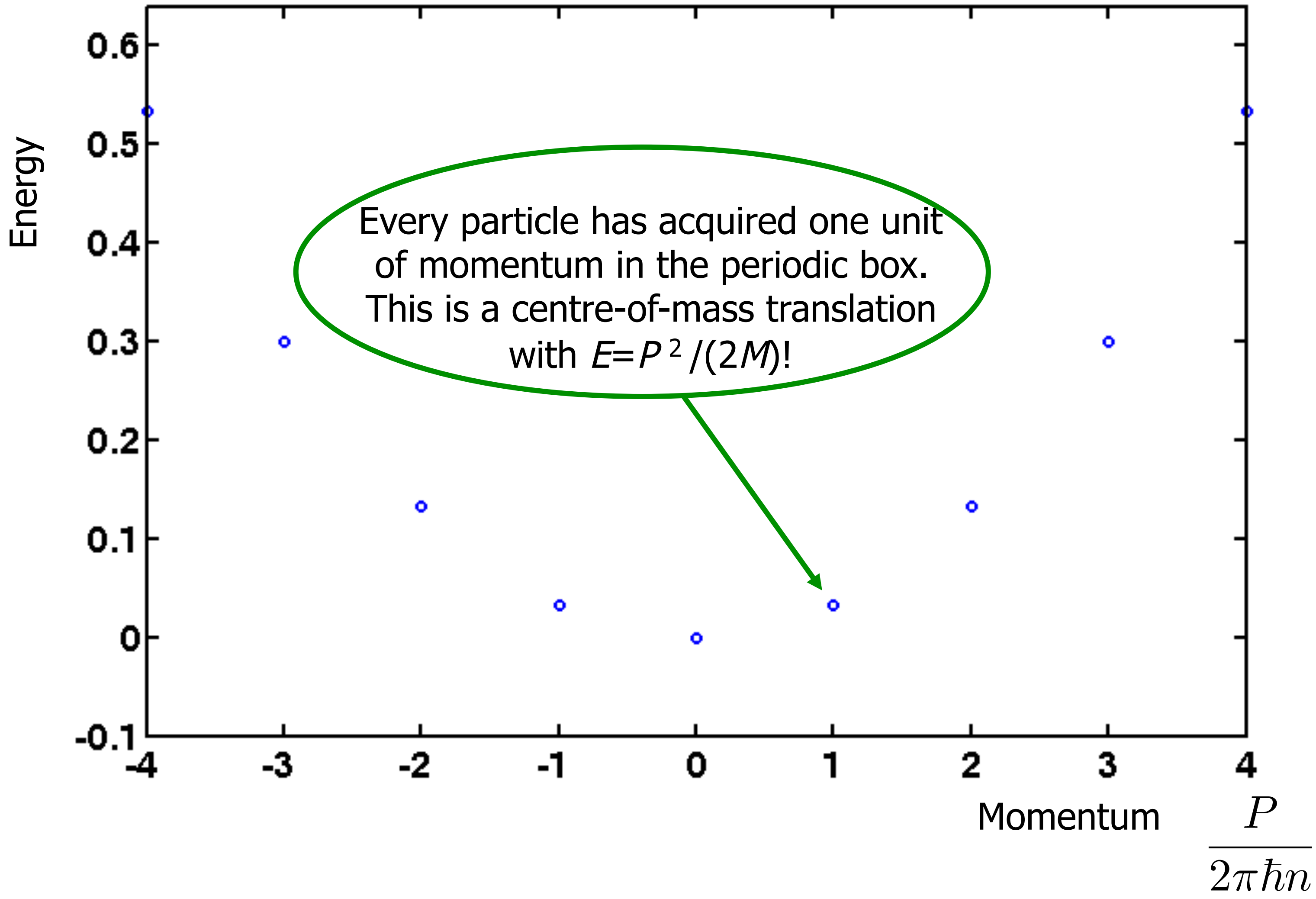
- Metastability of currents

  Once we have established a (ring) current: Is it stable? How and on which time scale will it decay to the ground state?

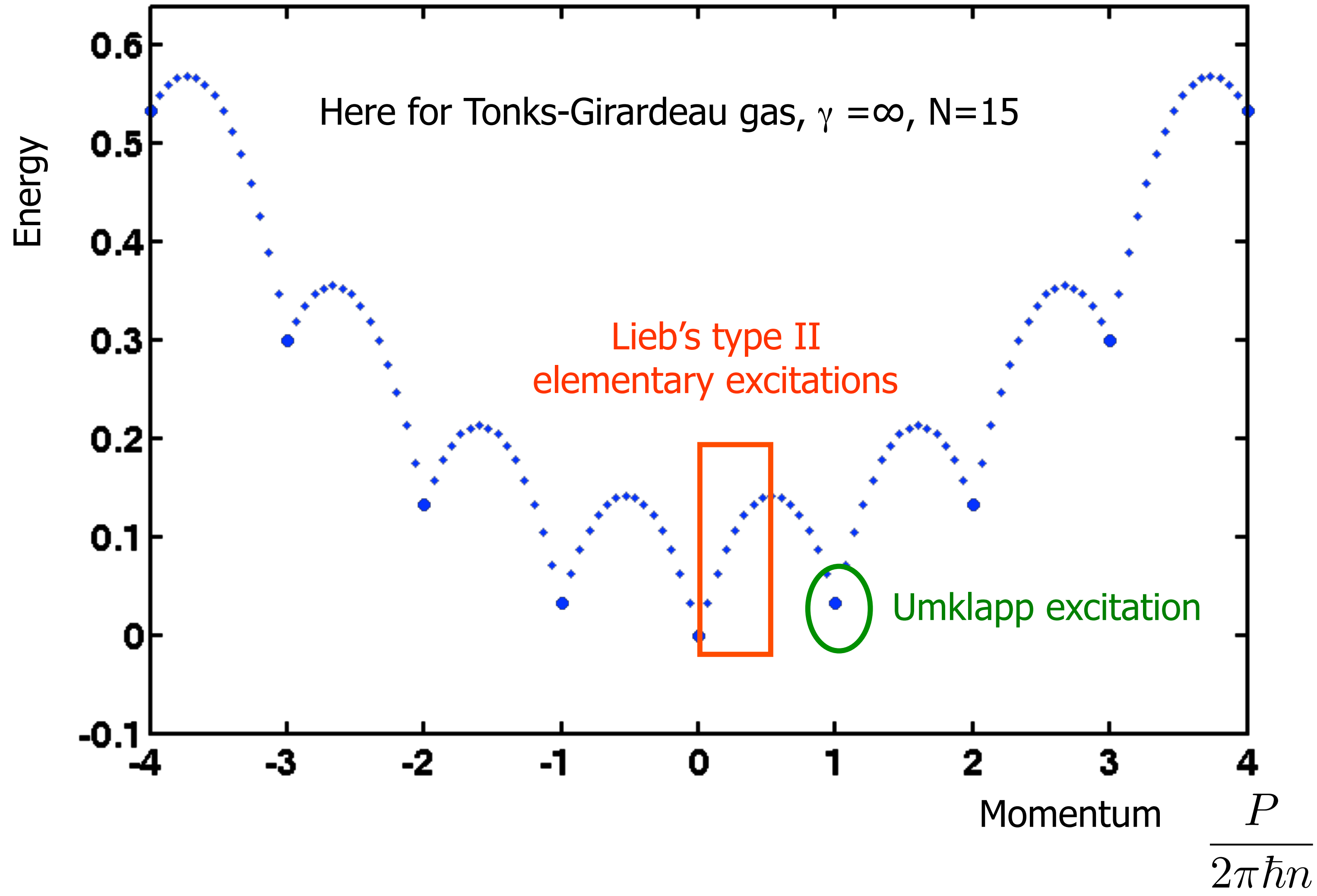
We need to understand the excitation spectrum and the likeliness of transitions!

Low-lying excitation spectrum

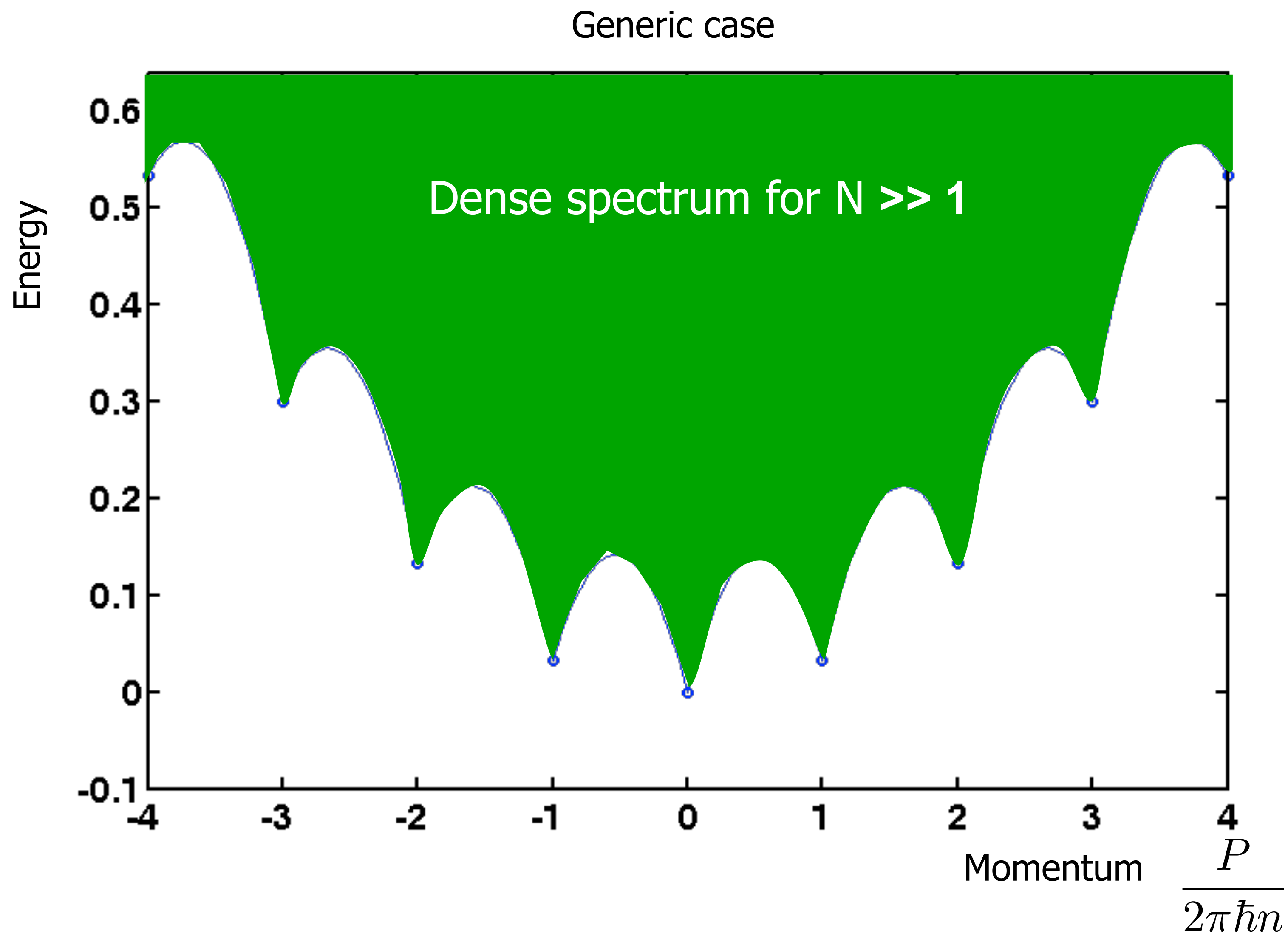
Translations of the ground state



Low-lying excitation spectrum (yrast states)

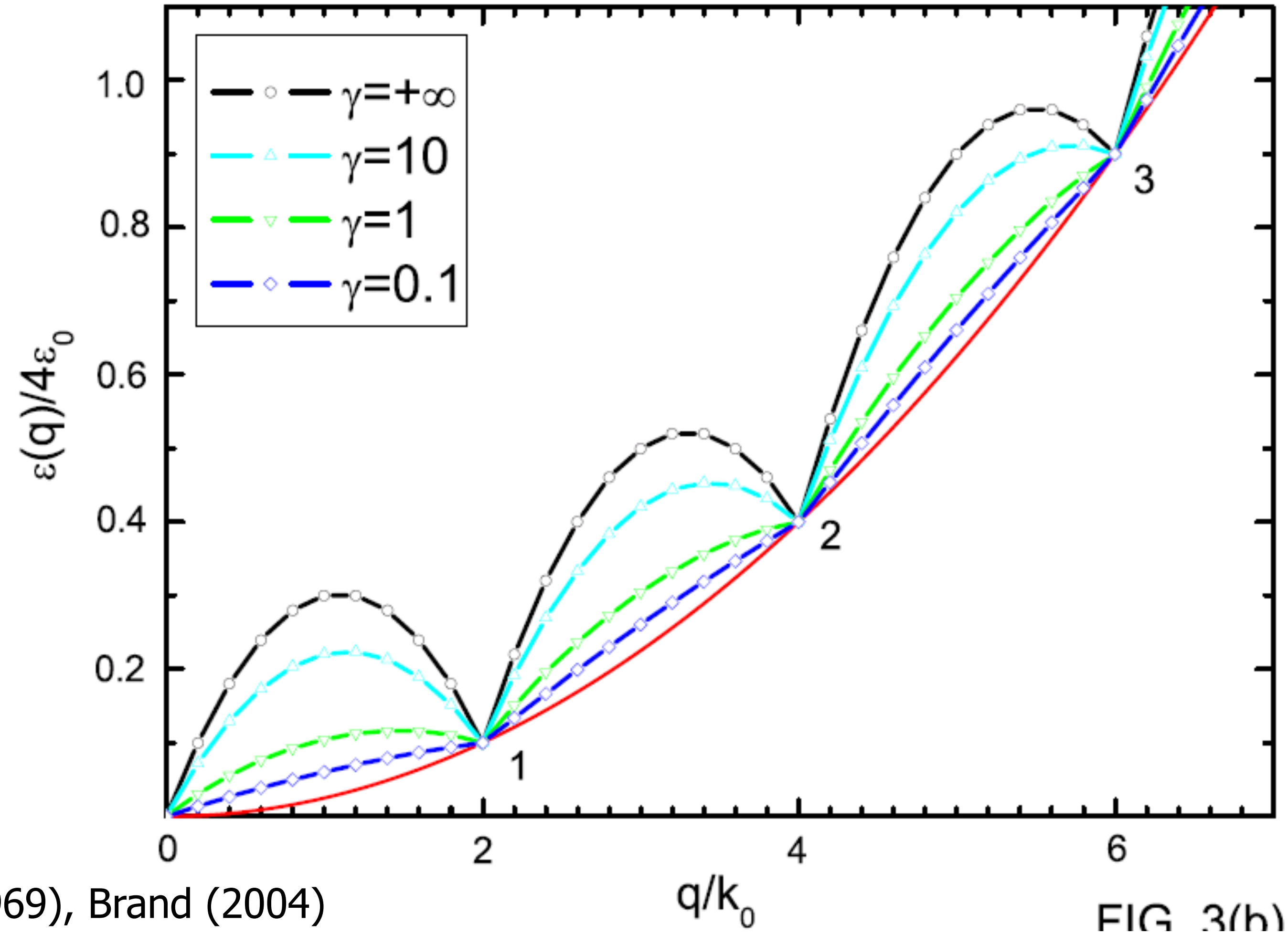


Low-lying excitation spectrum



Varying the interaction strength

The lowest energy excitation energies from the exact Bethe-Ansatz equations



Yang and Yang (1969), Brand (2004)

FIG. 3(b)

$k_0 = \pi \sim N/L$

Transformation to moving frame

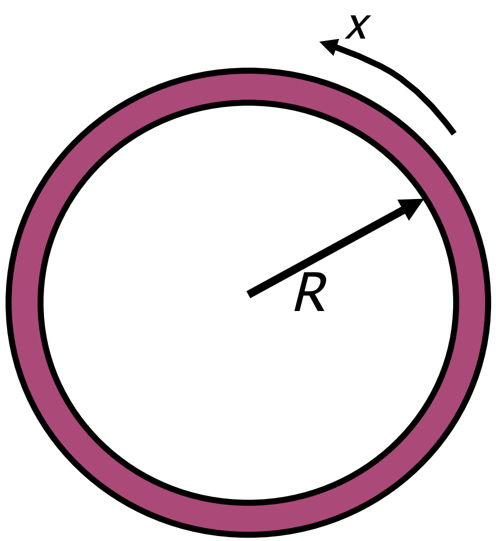
- A transformation into a translating (rotating) frame is a gauge transformation and does not change the internal nature of eigenstates.
- Momentum and energy change according to the well-known rules of **Galilean transformation**

$$P' = P + Mv_r$$

$$E' = E - Pv_r + \frac{1}{2}Mv_r^2$$

$$x' = x - v_r\, t$$

$$v_r = R\,\omega_r$$



Persistent currents in equilibrium

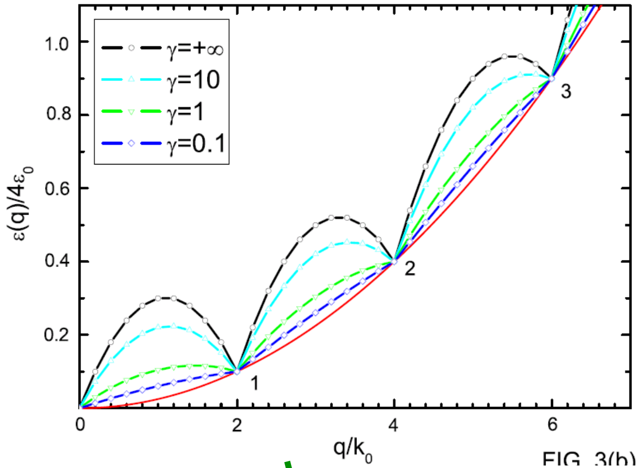


FIG. 3(b)

In the moving frame

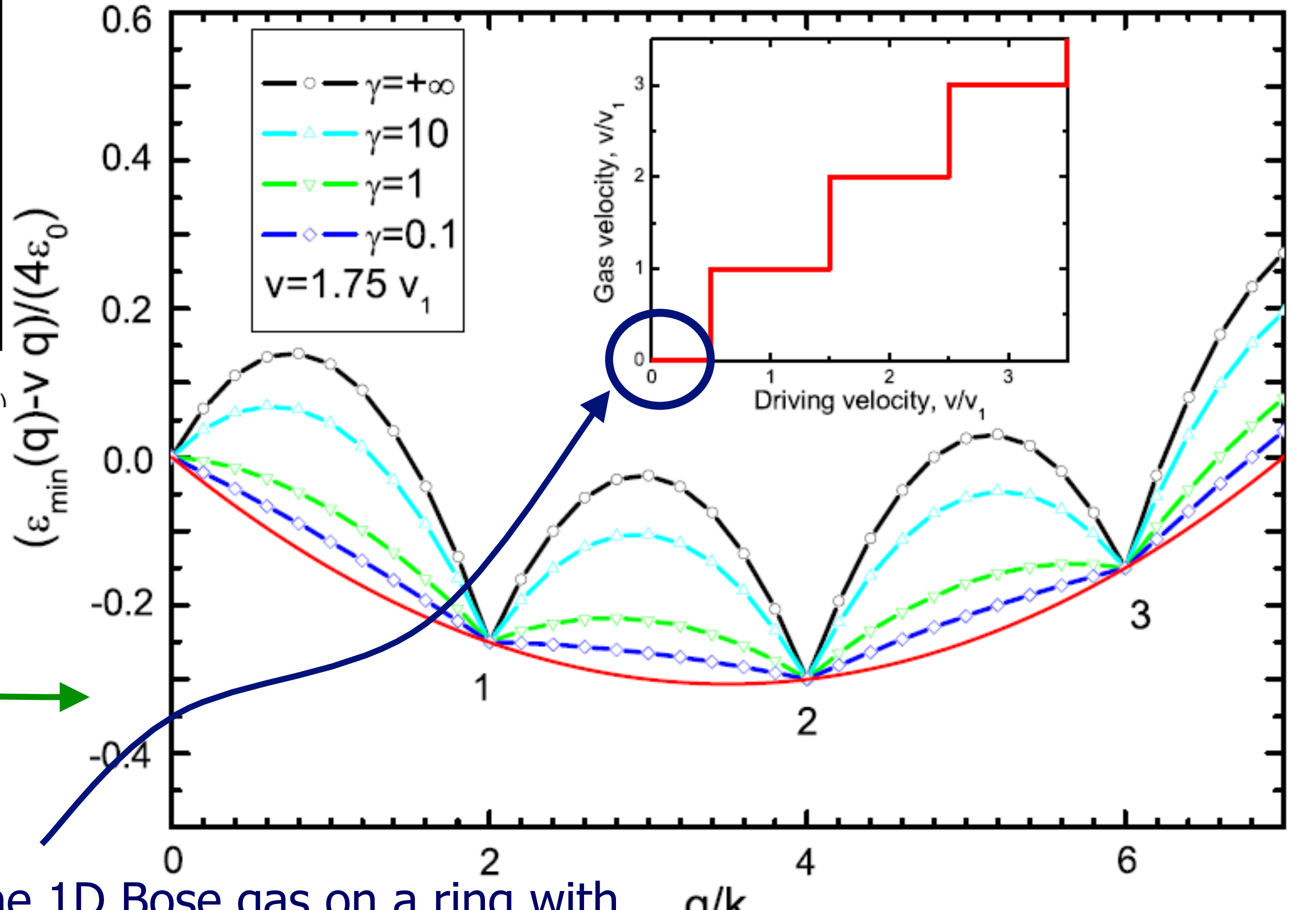


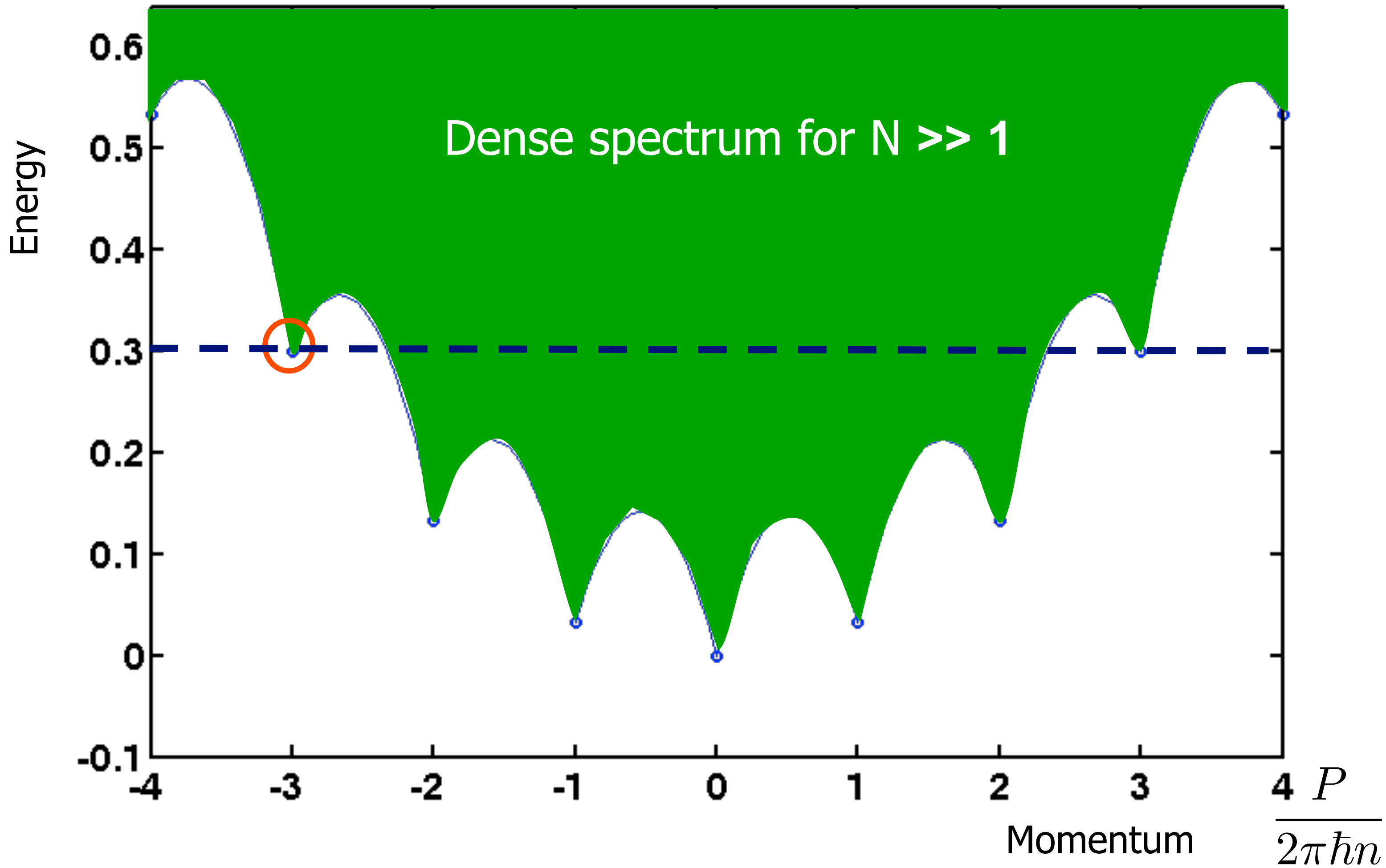
FIG. 3(c)

For small driving velocities the 1D Bose gas on a ring with repulsive interactions does not take part in the motion. This is known as the **Hess-Fairbank effect**!
Therefore it is **100% superfluid**.

$k_0 = \pi \sim N/L$

Metastability of currents

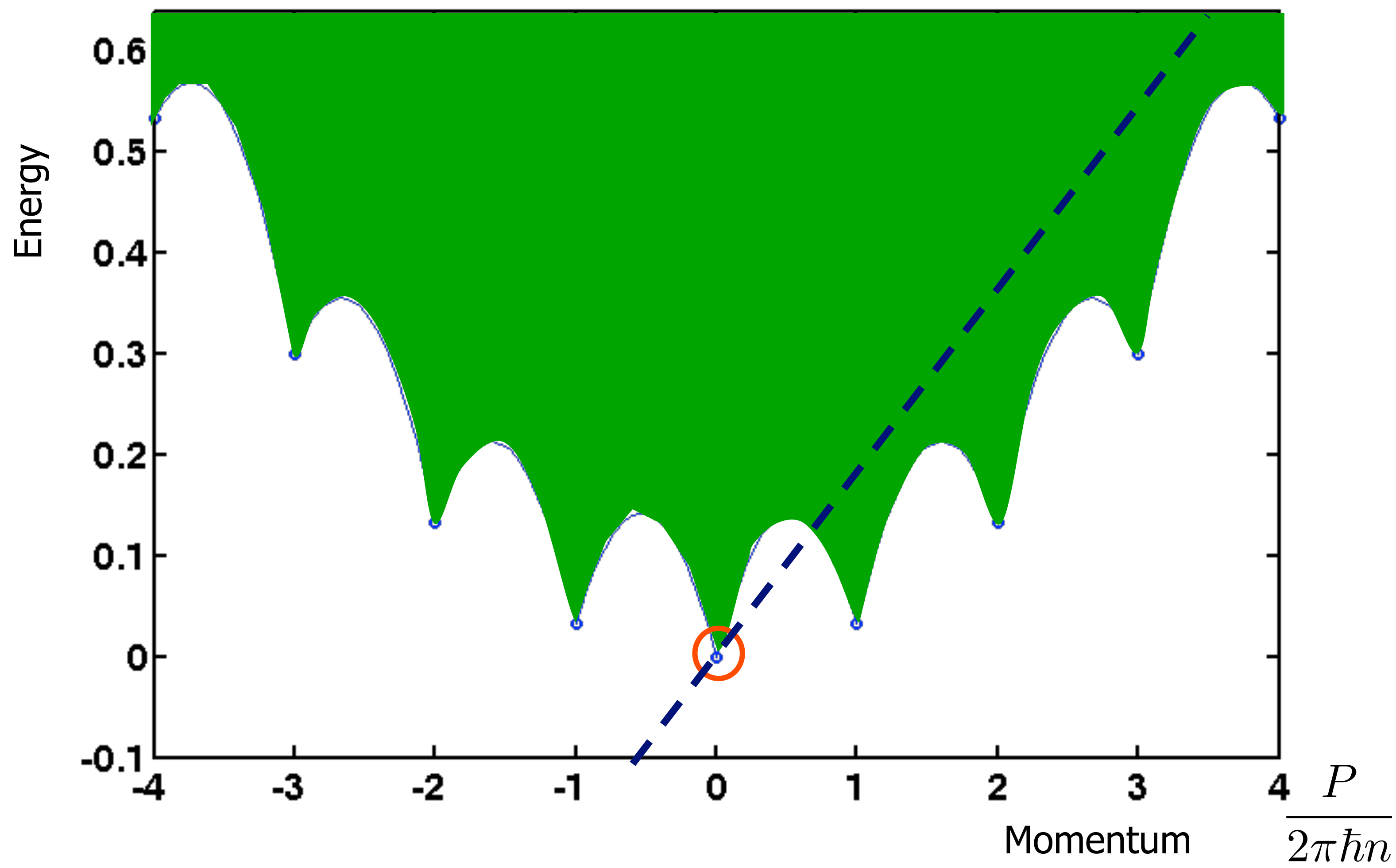
How can a current-carrying state decay?



A small deviation of the trapping potential (stationary in Lab frame) breaks translational symmetry and couples to lower momentum states (Energy is conserved)!

Frame transformation - frictional force

In the frame where the fluid is at rest, we have a moving impurity

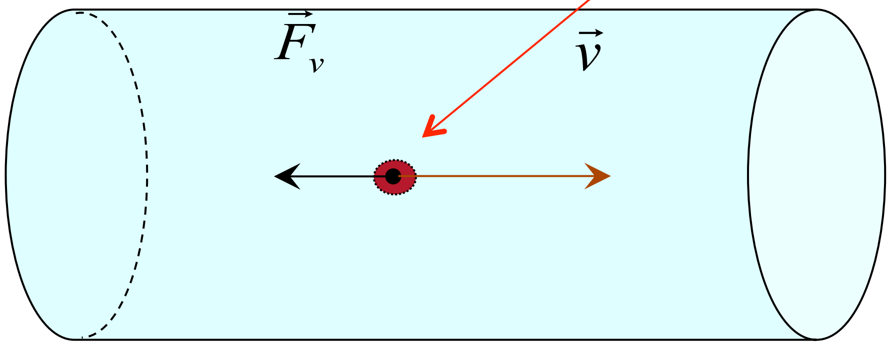


The moving impurity dissipates energy by exciting excitations in the fluid and experiences a **drag force**. The rate of dissipation can be calculated in linear-response theory

Drag Force

Consider a heavy impurity, moving with constant velocity in the 1D medium of particles and interacting with them by $V_{\mathsf{i}}(x,t) = g_{\mathsf{i}}\delta(x - vt)$

By definition $\dot{E} = -\mathbf{F}_{\mathrm{v}} \cdot \mathbf{v}$



Drag Force as a generalization of **the Landau criterion for superfluidity:** it should be zero to prevent energy dissipation!

$$F_v = 0 \quad ???$$

Drag Force

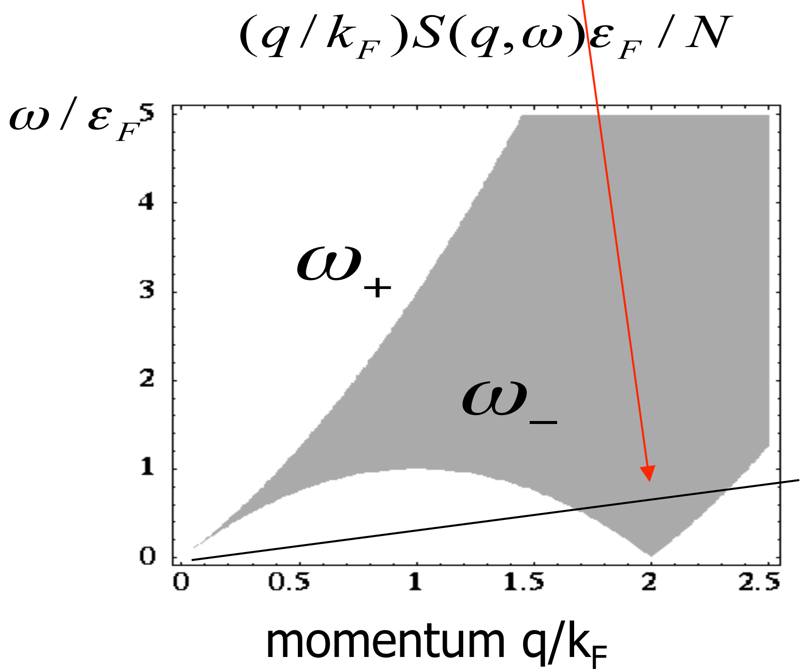
The linear response theory yields for the resulting drag force:

$$F_{\mathsf{v}} = \rho g_{\mathsf{i}}^2 \int_0^{+\infty} \mathsf{d}q\, q\, \frac{S(q, \omega = qv)}{N}$$

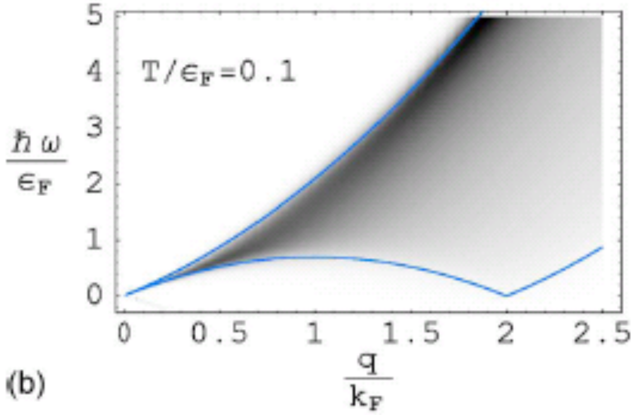
$$S(q,\omega) = \sum_n |\langle 0|\hat{\rho}_q|n\rangle|^2 \delta(\hbar\omega - E_n + E_0)$$

is the **dynamic structure factor**

Example:TG gas



Dynamic Structure factor



- What is known?
  - Limits of no interaction $\gamma = 0$
    and strong interaction $\gamma = +\infty$
    are well understood
  - Perturbation theory at large coupling
    PRA and first order expansion in $1/\gamma$

$$s(\lambda,\nu) = \frac{1}{4\lambda}\left(1+\frac{8}{\gamma}\right) + \frac{1}{2\gamma}\ln\frac{\nu^2-\nu_-^2}{\nu_+^2-\nu^2} + O\left(\frac{1}{\gamma^2}\right)$$

Brand and Cherny (2005)
Cherny and Brand (2006)

  - Luttinger liquid theory
    gives power-law scaling at small ω

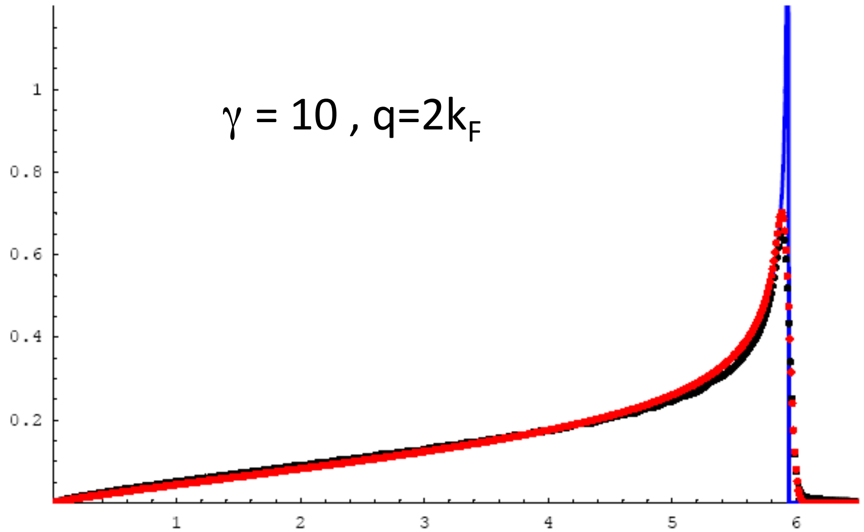
    Astrakharchik and Pitaevskii (2004)
  - Critical exponents near singularities

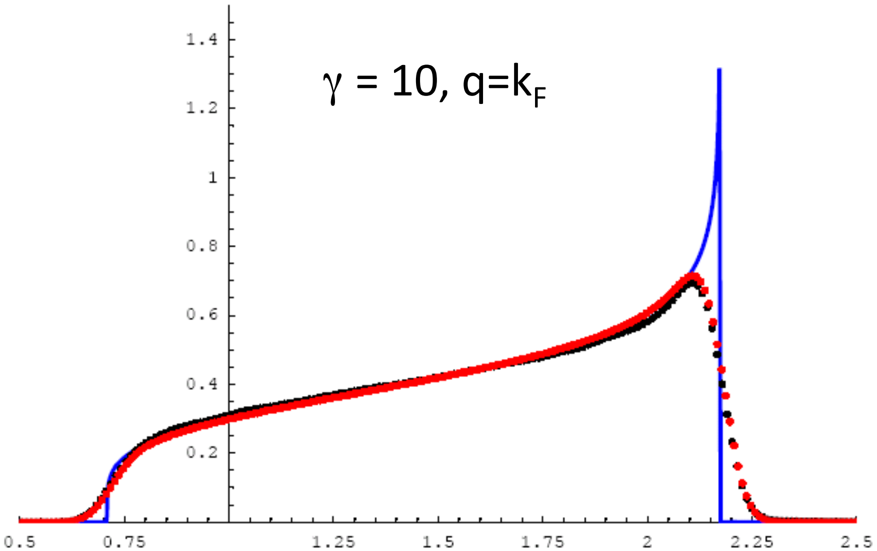
    Imambekov and Glazman (2008)
  - Numerical evaluation of the algebraic Bethe ansatz

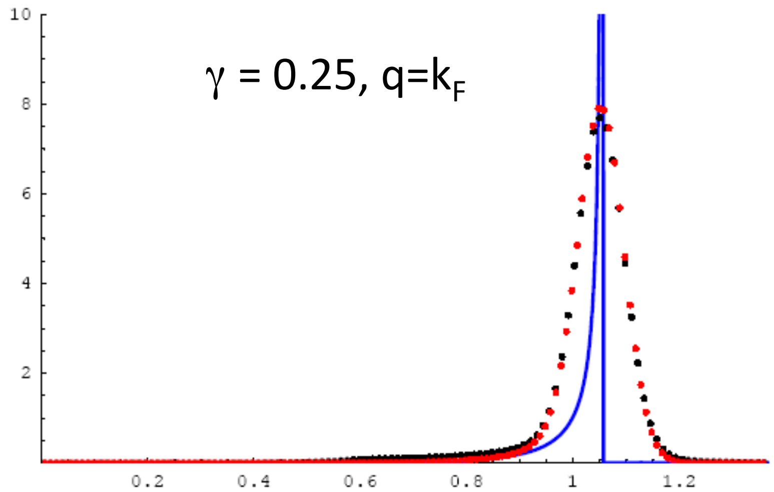
    Caux and Calabrese (2005)

Interpolating the Dynamic Structure Factor

$$s(\lambda,\nu) = 2\lambda^2 \frac{\Gamma(2+\mu_+ - \mu_-)}{\Gamma(1+\mu_-)\Gamma(1-\mu_+)} \frac{(\nu^2-\nu_-^2)^{\mu_-}}{(\nu_+^2-\nu^2)^{\mu_+}}$$





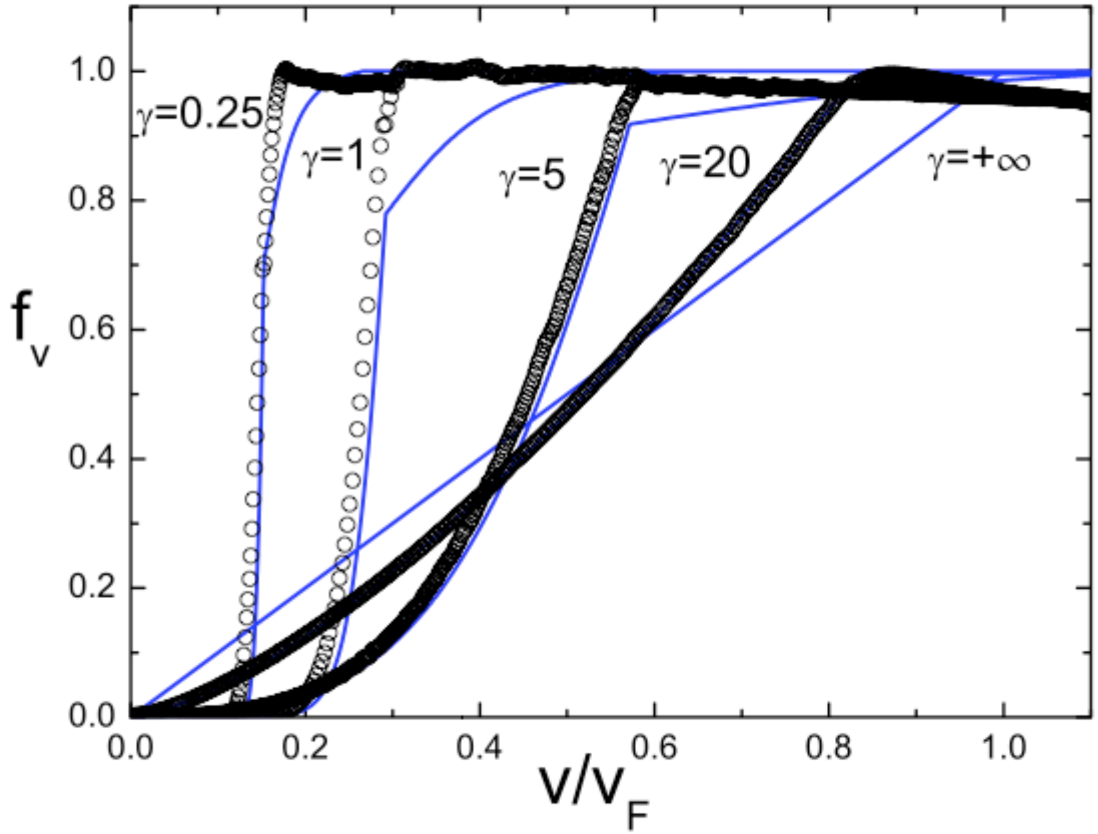


Cherny and Brand, PRA (2009); JP conf. ser. (2008)

Drag force = dissipation of supercurrent

Dimensionless drag force as a function of velocity

There is no critical velocity in the "Landau" sense!

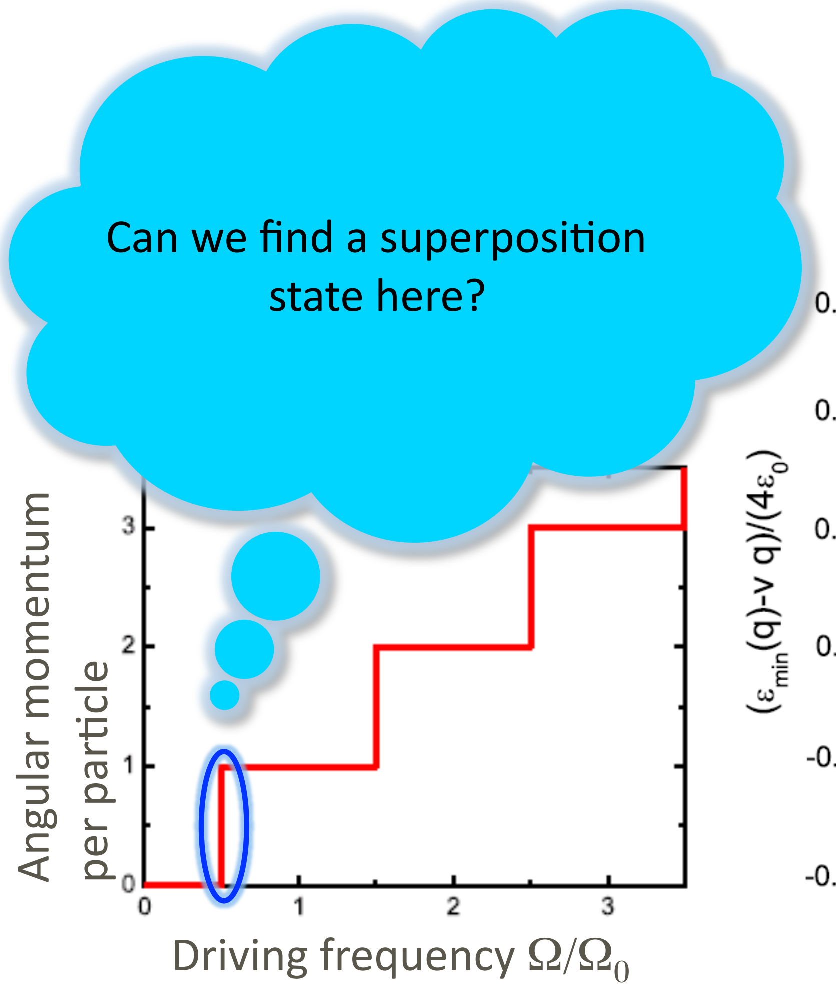


Only in the weak interaction (large density) limit $\gamma \to 0$ does the gas maintain persistent currents!

Cherny, Caux, and Brand, PRA (2010)

Schrödinger’s cat made robust

Possibility of superposition states?



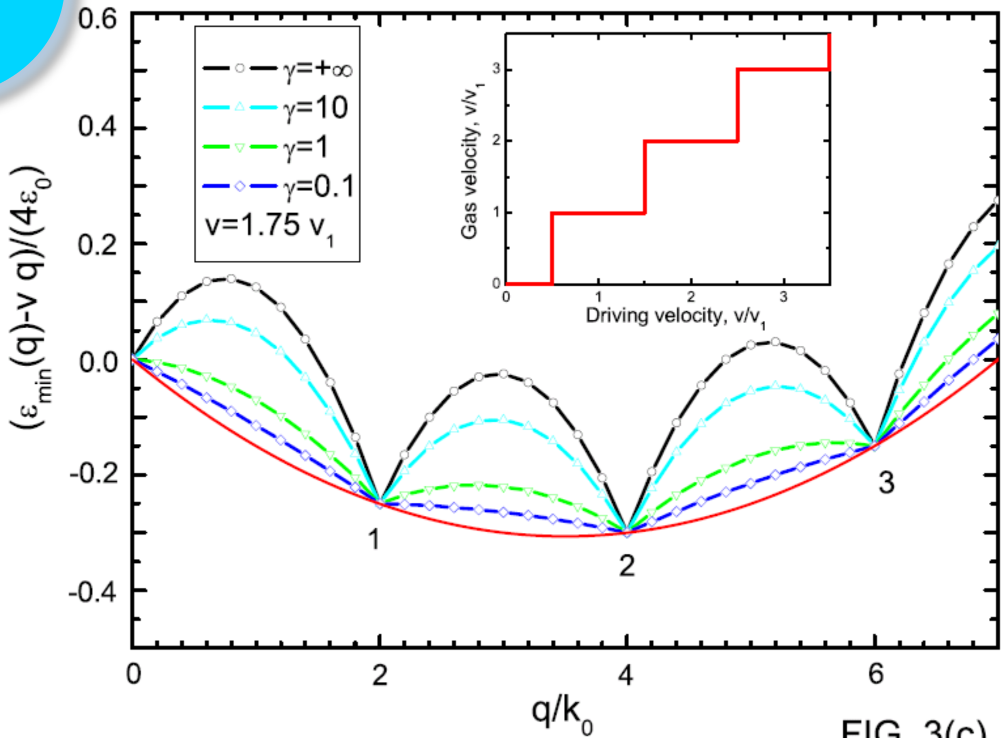
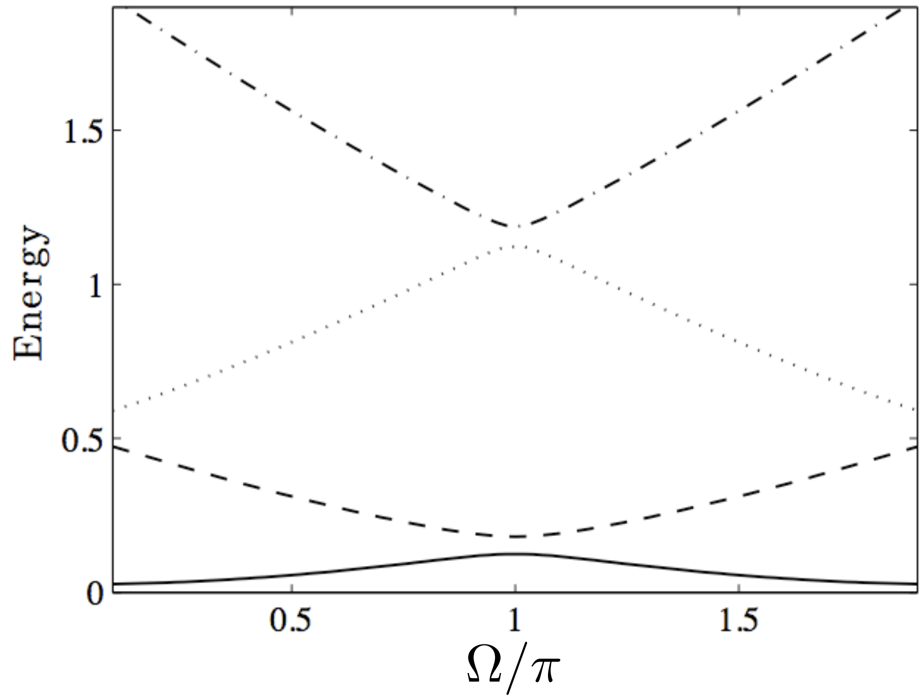


FIG. 3(c)

Rotating ring with delta barrier (cf. Josephson Junction)

$$H = \sum_{i=1}^{N} \left[\frac{\hbar^2}{2M} \left(-i \frac{\partial}{\partial x_i} - \frac{\Omega}{L} \right)^2 + \epsilon \delta(x_i) + g_1 \sum_{i<j}^{N} \delta(x_i - x_j) \right]$$

A single atom

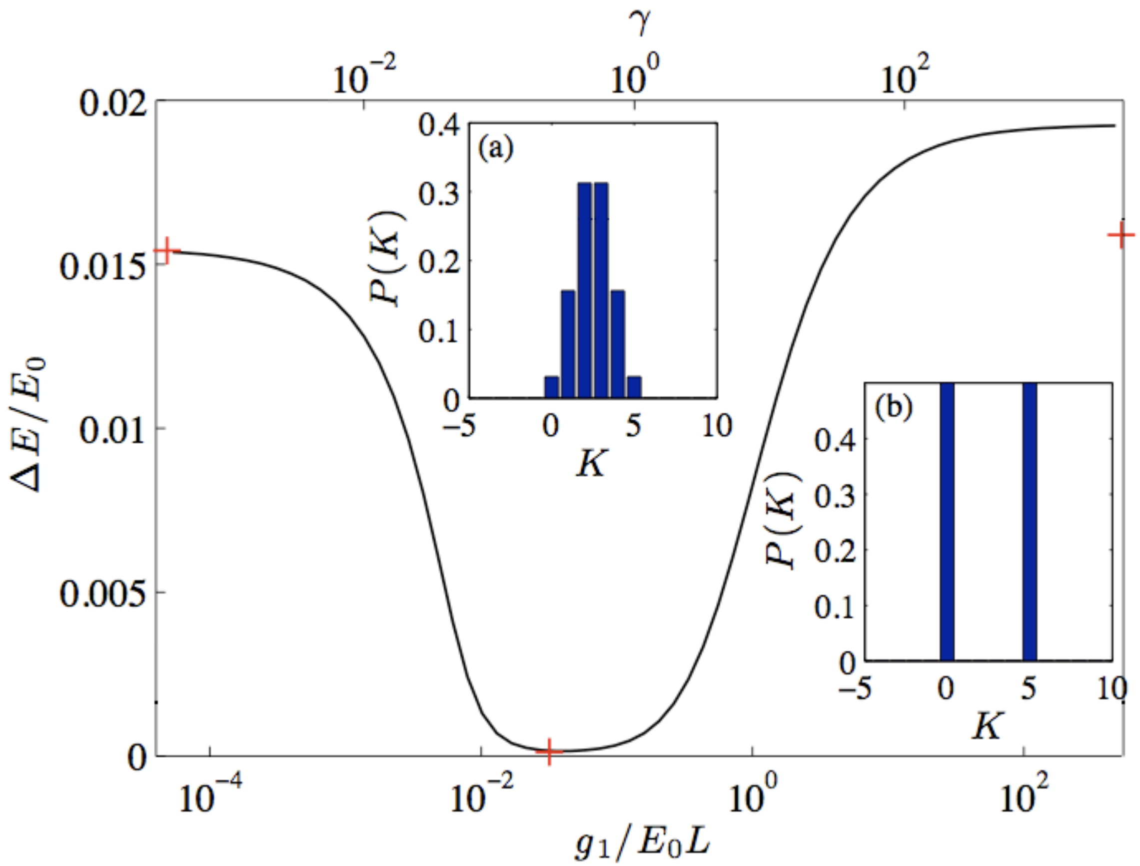


Many-body wave function for g=0 at $\Omega = \pi$

$$|\Psi_{\rm NI}\rangle \sim \prod_{i=1}^{N} \left(\phi_0(x_i) + \phi_1(x_i)\right)$$

Interacting particles: exact diagonalisation

Energy gap between ground and first excited state
as a function of interaction strength



N=5. Exact diagonalisation with 20 modes. $\varepsilon/L = 0.008E_0$

Hallwood, Ernst, Brand, PRA 2010

Weakly-interacting particles: NOON states

- Hamiltonian in momentum basis:

$$
\begin{aligned}
H_K &= \sum_k \frac{\gamma}{2} (2\pi k - \Omega)^2 \hat{a}_k^\dagger \hat{a}_k \\
H_B &= \epsilon \sum_{k_1,k_2} \hat{a}_{k_1}^\dagger \hat{a}_{k_2} \\
H_I &= \frac{g_1}{2L} \sum_{k_1,k_2,k_3,k_4} \hat{a}_{k_1}^\dagger \hat{a}_{k_2}^\dagger \hat{a}_{k_3} \hat{a}_{k_4} \delta(k_1 + k_2 - k_3 - k_4) \quad (7)
\end{aligned}
$$

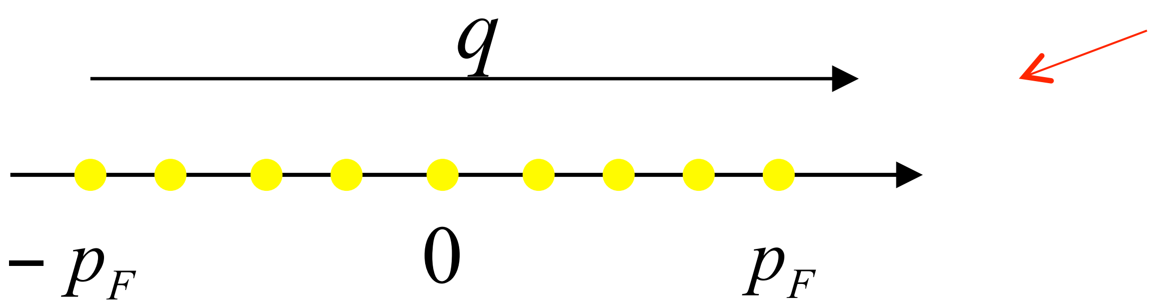
- For weak interactions, two modes suffice:
  NooN state has energy gap that scales exponentially with *N*!

$$
\left\{ (a_0^\dagger)^N + (a_1^\dagger)^N \right\} |\mathrm{vac}\rangle = |N, 0\rangle + |0, N\rangle
$$

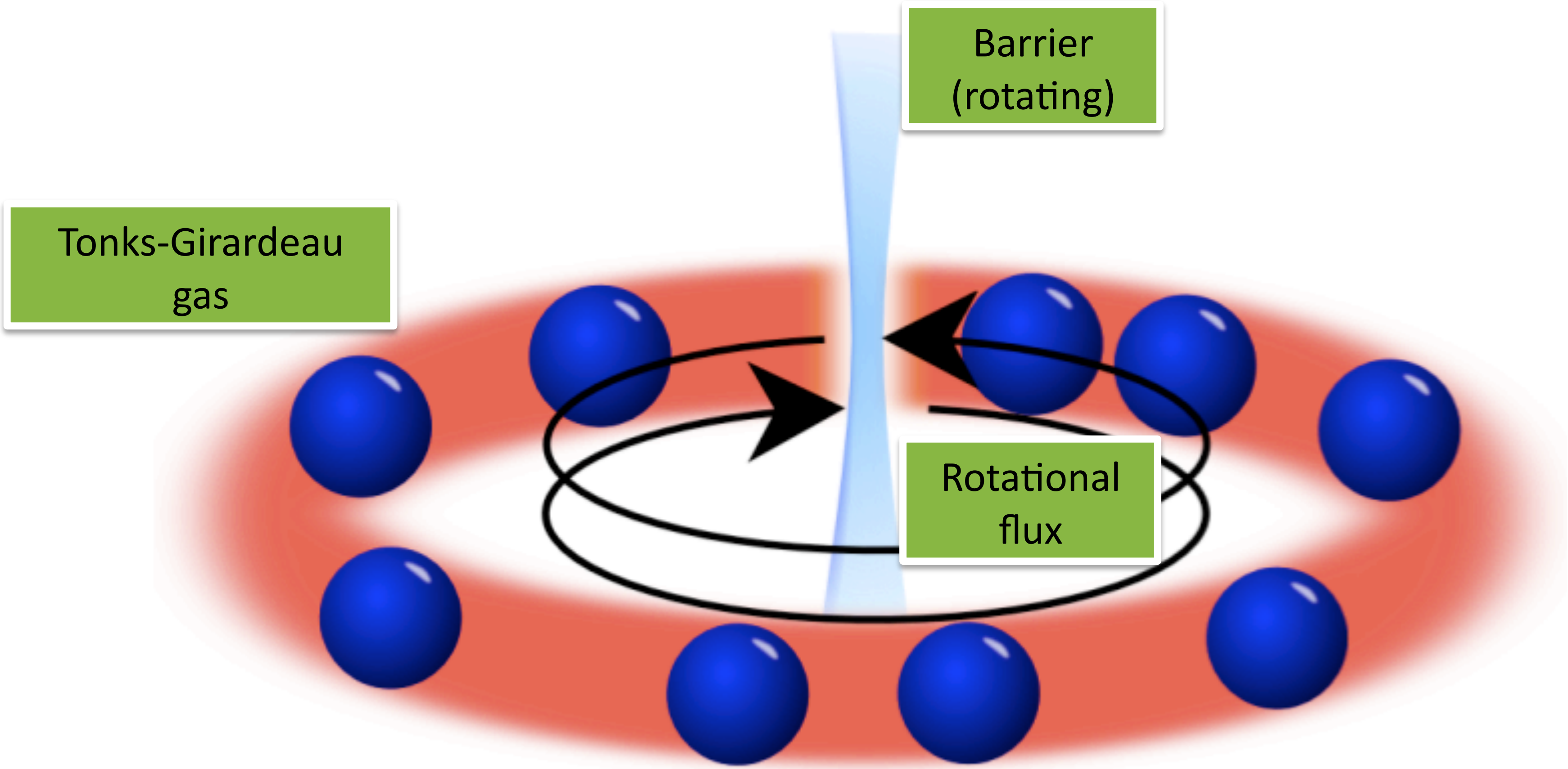
Strongly-interacting particles: Tonks-Girardeau gas

- The excitation spectrum is the same as that of non-interacting Fermions
- The gap energy is determined by the Umklapp excitation:

$\Delta E \approx \epsilon / L$ is independent of *N*!



Rotational Tonks-Girardeau cat



Hallwood, Ernst, Brand, PRA 2010

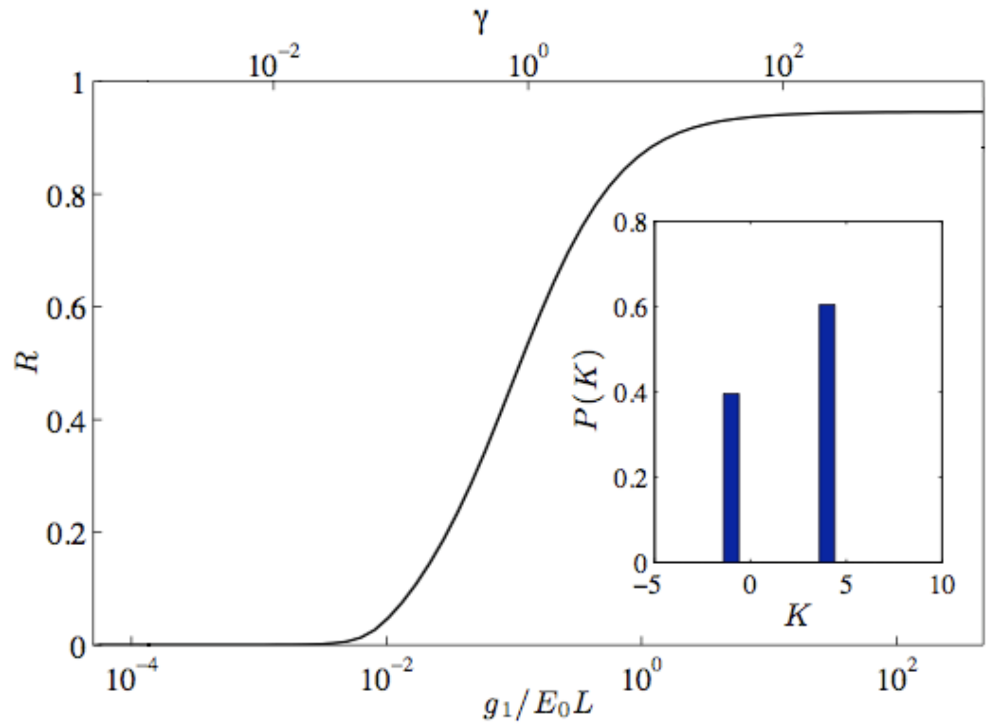
Tonks-Girardeau cats

- TG cats are robust against single-particle loss!
- Quality of cat after loss of one atom with momentum *k*:

$$Q_k = 2\sqrt{P(0-k)P(K-k)}$$

- Define *robustness* as the average

$$R = \sum_k Q_k n_k / N$$



Hallwood, Ernst, Brand, PRA 2010; Cooper, Hallwood, Dunningham, Brand, PRL 2012

Stirring up solitons

Can you wind/unwind a ring current by following the type II dispersion curve?

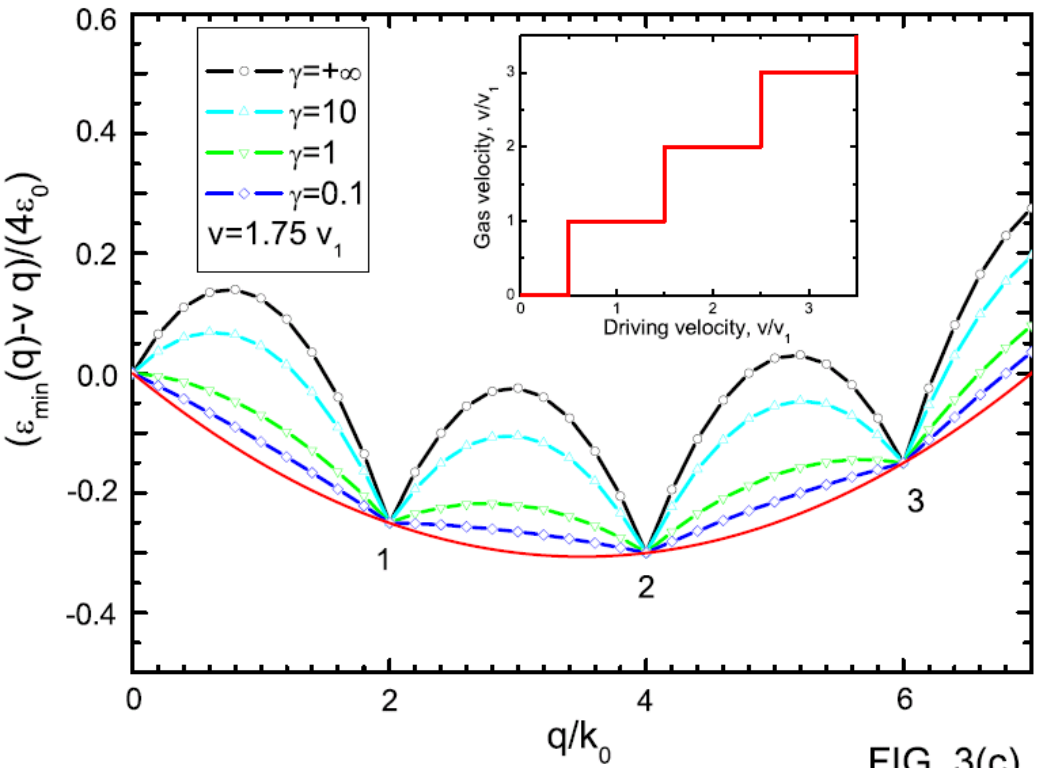
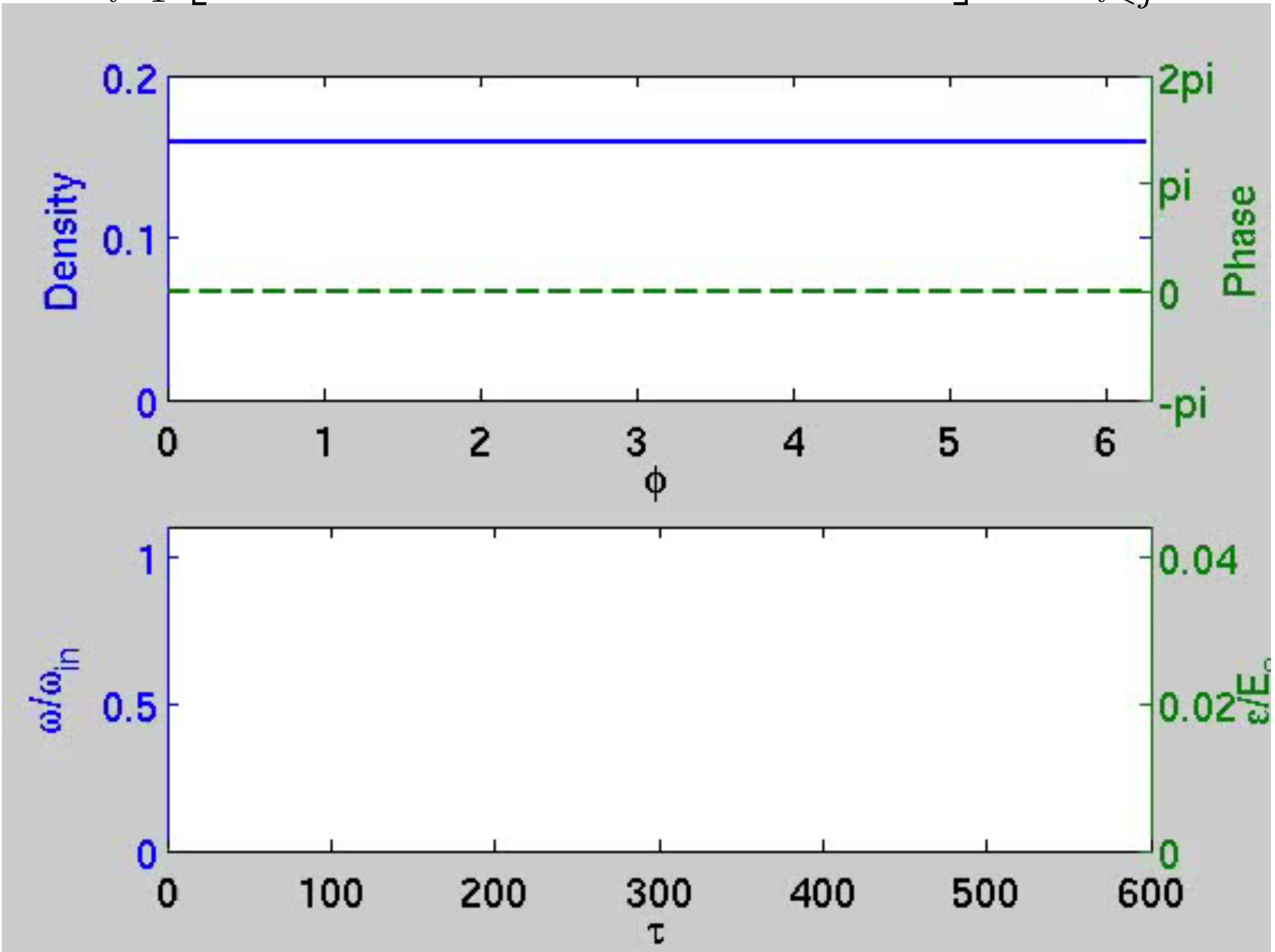


FIG. 3(c)

Kanamoto, Carr, Ueda, PRL 2008, PRA 2009, 2010

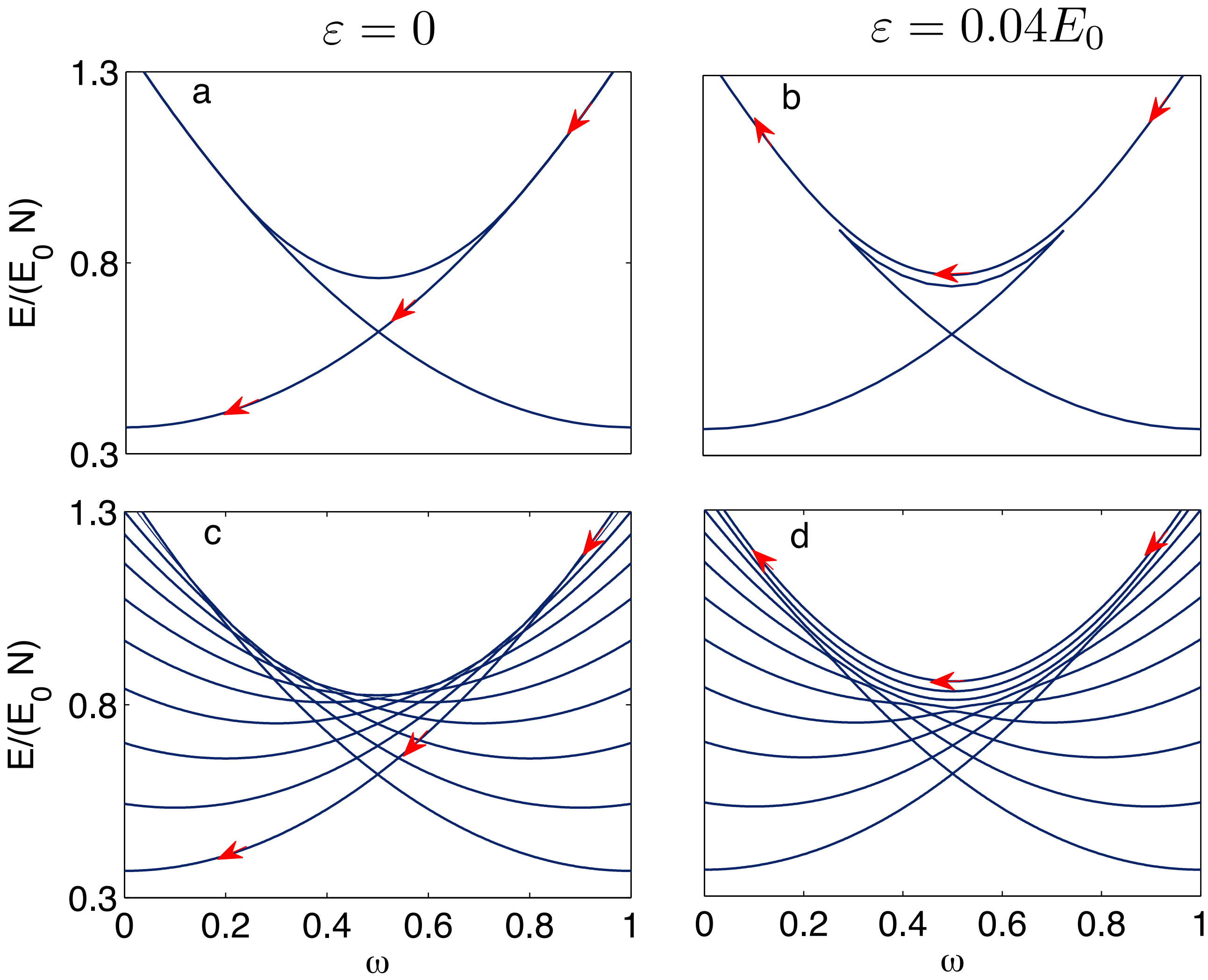
Add a global symmetry-breaking potential

$$H = \sum_{i=1}^{N} \left[\frac{\hbar^2}{2m} \left(-i \frac{\partial}{\partial x_i} - \frac{\omega}{L} \right)^2 + \epsilon \cos\left(\frac{2\pi x_i}{L} \right) \right] + g_1 \sum_{i<j}^{N} \delta(x_i - x_j)$$



Fialko, Coralie-Delattre, Brand, Kolovsky 2012

Adiabatic passage through metastable states



One dimension is different!

- To be covered in these lectures:
  - Absence of true Bose-Einstein condensation
  - Strongly-correlated many-body physics with a dilute gas
  - Bosons play fermions
  - Superfluid or not superfluid (or maybe both?)
  - Schrödinger cats made robust
  - Stirring up solitons

Bibliography

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