Quantum Entanglement

Victorian Summer School

M. Reid
Swinburne University of Technology, Melbourne, Australia

Outline

1. Non-locality and quantum mechanics

Einstein's (EPR) spooky action at a distance 1935
Schrodinger's cat 1935
Bell's theorem 1965
- Bell and EPR experiments

GHZ's extreme multiparticle quantum nonlocality

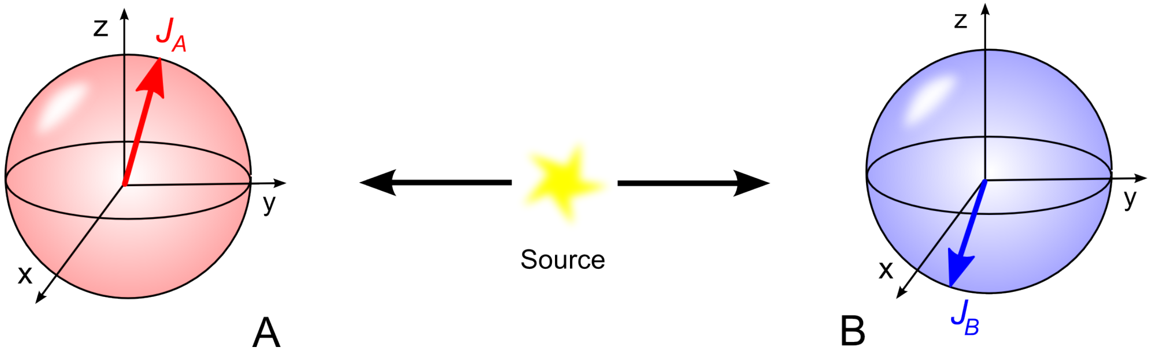
2. Introduce formalism of entanglement

Density operator – mixed states
Inseparability of density matrix
Pauli spin examples
Werner states
Peres PPT criterion and concurrence
Quadrature squeezing and spin squeezing
CV Variance and spin squeezing criteria for entanglement

3. Applications

Quantum cryptography and quantum teleportation

So - Schrodinger's Entangled States



$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

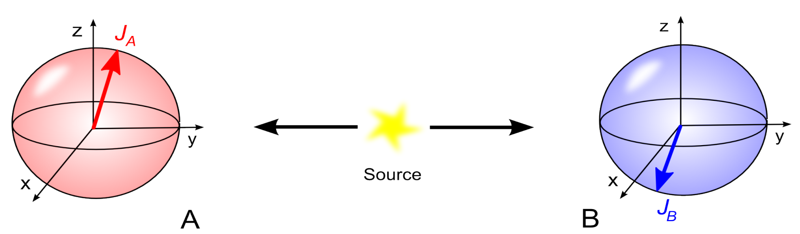
A **(pure) entangled** state is one that cannot be written in any factorised form i.e.

$$|\phi\rangle \neq |\psi_A\rangle|\psi_B\rangle$$

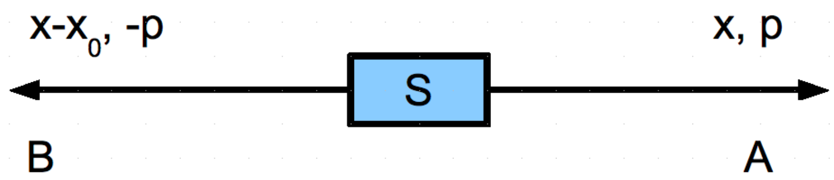
Entangled states: let's look at them

Entangled states are non-separable: 2 classic examples

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$



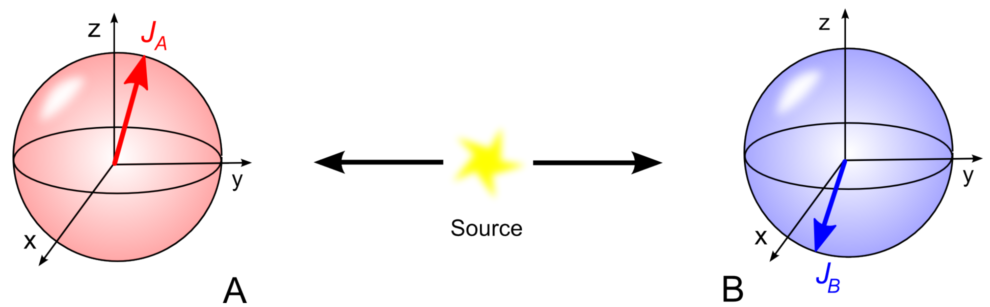
$$\delta(x_A - x_B)\delta(p_A + p_B)$$



•Entangled states - greater correlation than separable states for *both conjugate* (non-commuting) observables

Are entanglement and nonlocality equivalent?

NONLOCALITY requires entanglement

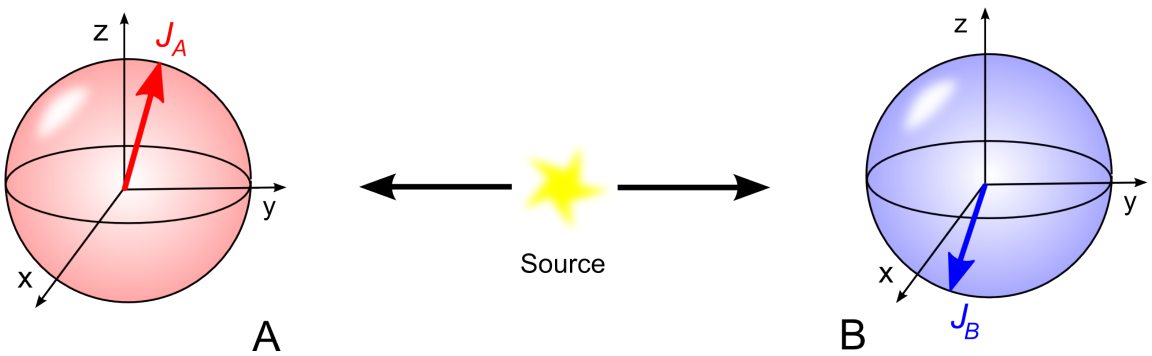


Interesting results for two qubit case

- All 2 qubit pure entangled states violate CHSH Bell inequality (Gisin)

- BUT there exist states (Werner) that are entangled
  but are consistent with Local realism
  ie cannot violate a Bell inequality
- Answer - no

So - Schrodinger's Entangled States



$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

A **(pure) entangled** state is one that cannot be written in any factorised form i.e.

$$|\phi\rangle \neq |\psi_A\rangle|\psi_B\rangle$$

However, not all states are pure states!

Quantum entanglement:
First we revise some QM formalism

For any pure state $$|\psi\rangle$$

the **density operator** is defined

$$\rho = |\psi\rangle\langle\psi|$$

Need some formalism for density operators ρ

Look at notes on line:

Recall: A density operator for a pure state $|\psi\rangle$ is the *operator* $|\psi\rangle\langle\psi|$ – here we use bra - ket notation.

So consider spin 1/2 system.

Question: Suppose the system is in $|\psi\rangle = |\uparrow\rangle \equiv |1/2, 1/2\rangle$ – what is ρ? :

Answer: Express ρ in spinor basis $|1\rangle \equiv |\uparrow\rangle$, $|2\rangle \equiv |\downarrow\rangle$, so $\rho_{ij} = \langle i|\rho|j\rangle$

$$\rho \;=\; \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Question: What is ρ for superposition? $\quad |\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A - |\downarrow\rangle_A)$

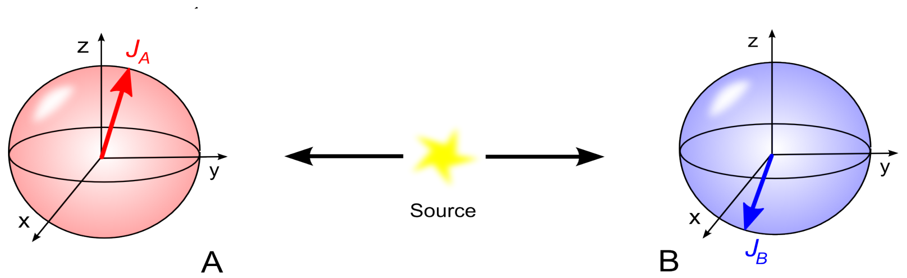
Answer:

$$\rho \;= \tfrac{1}{2} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

coherences

Note the off-diagonal elements associated with the superposition!

For the Bell state?



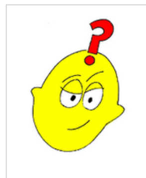
Question: What is the ρ for the singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A |\downarrow\rangle_B - |\downarrow\rangle_A |\uparrow\rangle_B) \ ?$$

What is that basis?

Answer: take suitable basis

$$\rho = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$



What is ρ for the 50/50 mixture $|\uparrow\rangle_A|\downarrow\rangle_B$ and $|\downarrow\rangle_A|\uparrow\rangle_B$?

And for the mixture?

Question: What is the ρ for the singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B) \ ?$$

Answer: take suitable basis

$$\rho \;=\; \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

New question

What is ρ for the 50/50 mixture $|\uparrow\rangle_A|\downarrow\rangle_B$ and $|\downarrow\rangle_A|\uparrow\rangle_B$?

To answer, we introduce the definition:

Definition: The density operator for a mixed state ie a state that is in a mixture of pure states $|\psi_R\rangle$ with probability P_R is given as

$$\rho = \sum_R P_R|\psi_R\rangle\langle\psi_R|$$

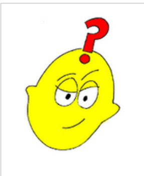
Need some formalism: mixed states density operators ρ

Question: What is the density matrix for the 50/50 mixture of spin 1/2 $|\uparrow\rangle$ and spin $-1/2$ $|\downarrow\rangle$ states?

$$\rho = \frac{1}{2}[|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|]$$

Answer:

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$



Now answer our question

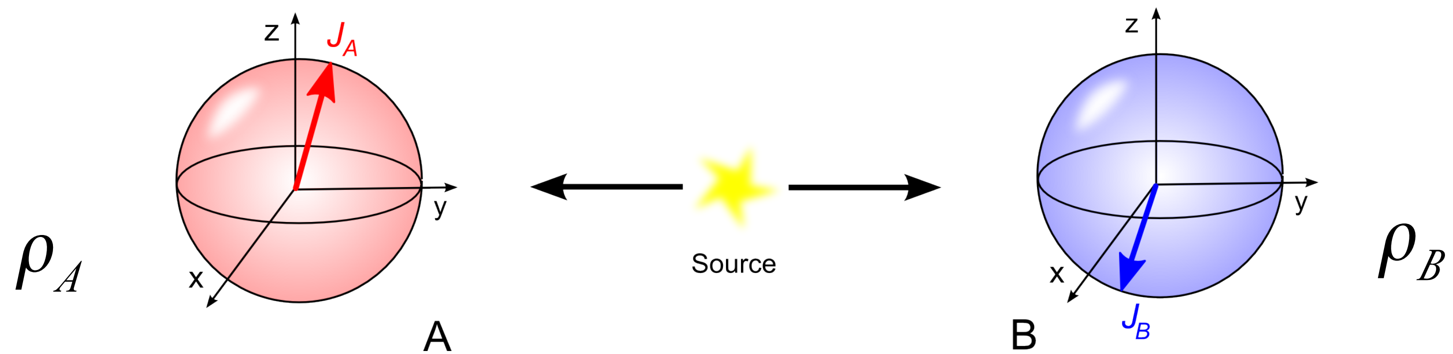
Exercise: Now evaluate density operator and matrix for the 50/50 mixture of $|\uparrow\rangle_A|\downarrow\rangle_B$ and $|\downarrow\rangle_A|\uparrow\rangle_B$?

Now evaluate matrix ρ for a spin 1/2 system in a 50/ 50 mixture of two superpositions:

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\downarrow\rangle),\ \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle).$$

Exercise 7: Prove for all pure states, that $\rho^2 = \rho$. Hence show that for pure state $P = Tr(\rho^2) = 1$. For a mixture $P < 1$.

Definition: SEPARABLE Quantum States



P_R probability

ρ density operator

$$\rho = \sum_R P_R \rho_A^R \rho_B^R$$

A state is separable iff ρ can be written as mixture of product states

Is $|\uparrow\rangle_A |\downarrow\rangle_B$ separable? $\frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B\right)$?

Definition of ENTANGLEMENT

2.3 Entanglement

We say two systems A and B are separable iff we can express the density operator in the following *factorisable* form:

$$\rho = \sum_R P_R \rho_R^A \rho_R^B$$

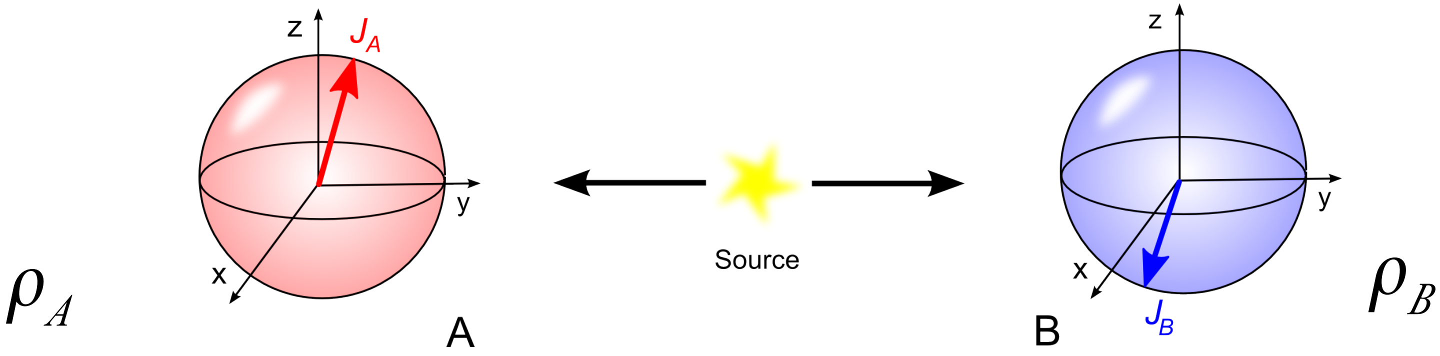
where ρ_R^A and ρ_R^B are density operators for system A and B respectively. If this cannot be done, we say the two systems are *inseparable* or *entangled*.

Exercise: Take the system 50/ 50 mixture of singlet and triplet superpositions:

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle),\ \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle).$$

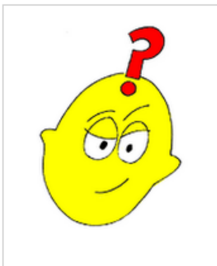
Is it entangled?

When do we lose entanglement?



$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B\right)$$

Take the Bell state, and add "noise"
ie consider a mixture of the entangled state with a noisy unentangled state



Werner state entanglement

Exercise: Consider the "maximally unentangled state" for two spin 1/2 systems (two "qubits"). This is a system in a equal mixture of the composite spin eigenstates.

What basis are we using?

$$\rho_{noisy} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{4}\mathbf{I}$$

Checkpoint: is this state Indeed not entangled?

Consider the system in a mixture of this maximally unentangled state ρ_{noisy} and the Bell singlet state:

$$\rho = p\rho_{singlet} + (1-p)\rho_{noisy}$$

Question: For what p is this state (called a Werner state) entangled?

(This is a fundamental question without an obvious immediate answer- see Peres, Physical Review Letters, method of Positive Partial Transpose PPT)

Positive Partial Transpose PPT condition for entanglement

Exercise: Peres PPT criterion (PRL, 77, 1413 1996)
We write density matrix for Werner mixed state as

$$\rho \;=\; \begin{pmatrix} (1-p)/4 & 0 & 0 & 0 \\ 0 & (p+1)/4 & -p/2 & 0 \\ 0 & -p/2 & (p+1)/4 & 0 \\ 0 & 0 & 0 & (1-p)/4 \end{pmatrix}$$

Then write the partial transpose wrt one system only

$$\rho \;=\; \begin{pmatrix} (1-p)/4 & 0 & 0 & -p/2 \\ 0 & (p+1)/4 & 0 & 0 \\ 0 & 0 & (p+1)/4 & 0 \\ -p/2 & 0 & 0 & (1-p)/4 \end{pmatrix}$$

Evaluate eignevalues of this new matrix.

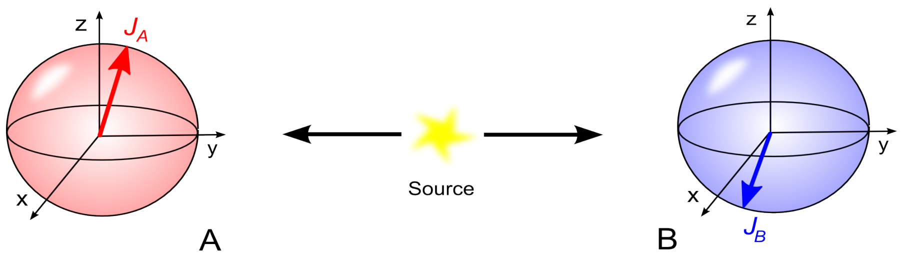
If the system is separable (not entangled) then all eigenvalues will be nonnegative. (Peres positive partial transpose condition-PPT).

All eigenvalues are nonnegative except one which is

$$\lambda = -(3p-1)/4$$

so if $p > 1/3$ we definitely have entanglement

PPT condition necessary and sufficient for entanglement for 2 x 2 systems



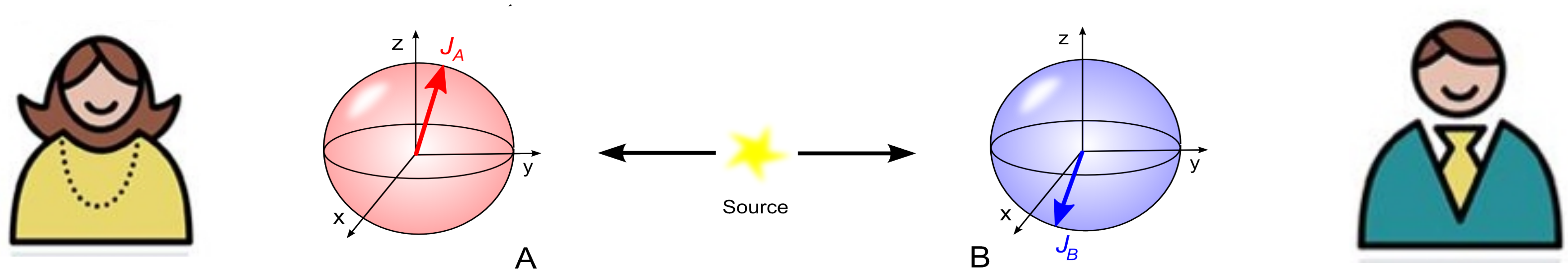
For 2 qubit systems, PPT criterion is necessary and sufficient

For systems of dimension 2x2 (ie spin 1/2 by spin 1/2), the PPT criterion is necessary and sufficient for entanglement - this was proved later by the Horodecki's (Physics Letter A223, 1, 1996; PRL78, 574 1997; 80,5239,1998). Thus there is no advantage in using the method of Wootter's concurrence which also gives you a necessary and sufficient condition for entanglement but only for 2x2 systems.

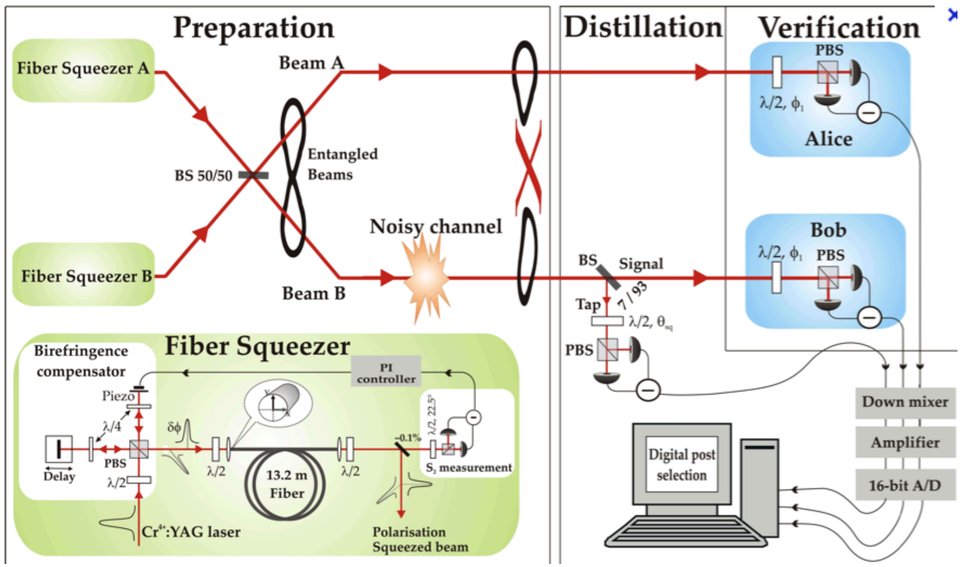
Hence we have separability when $p \leq 1/3$ and entanglement when $p > 1/3$.

For higher dimensional systems, it is possible to have states with positive partial transpose that are entangled- this is called BOUND entanglement.

BOUND entanglement cannot be "distilled"



"Distillation" means taking the "noisy" EPR entangled pairs, and selecting a smaller number of better entangled pairs USING only "local" operations and "classical communication" between Alice and Bob



Entanglement measures

How do we measure entanglement? *Requirements?*
Not made bigger by "local operations" or "classical communications"
Entanglement of a mixture can't exceed sum of the entanglement of its parts (convexity)

Pure state: *Entropy of entanglement*

One measure of entanglement is the von Neumann entropy S_A of the reduced matrix $\rho^A = \mathrm{Tr}_B|\Phi\rangle\langle\Phi|$

$$E(|\Phi\rangle\langle\Phi|) = S_A = S_B = -\mathrm{Tr}\rho^A \log \rho^A = -\sum_i \lambda_i \log \lambda_i$$

where λ_i are eigenvalues of ρ^A

Exercise: Take the Bell singlet $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B)$. What is the reduced density matrix for Alice's system? What is its E?

$$\rho^A = \frac{1}{2}[|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|]$$

Entanglement measures

Pure state: *Entropy of entanglement*

Alternatively, if we can write the state in the Schmidt basis as

$$|\Phi\rangle = \sum_{i=1}^{N} c_i |\psi_i\rangle_A \otimes |\psi_i\rangle_B$$

The measure of entanglement is

$$E(|\Phi\rangle\langle\Phi|) = S_A = S_B = -\sum_i |c_i|^2 \log|c_i|^2$$

If $c_1=1$, $E=0$, therefore there is no entanglement.

If $c_i = \frac{1}{\sqrt{N}}$ we have the maximum entanglement

Exercise: Take the Bell singlet $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B)$. What is the reduced density matrix for Alice's system? What is its E?

$$\rho^A = \frac{1}{2}[|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|]$$

The Bell state is maximally entangled!

Mixed state entanglement measures

Mixed states $\rho = \sum_R P_R |\psi_R\rangle\langle\psi_R|$ are more difficult. One measure is the *Entanglement of formation*:

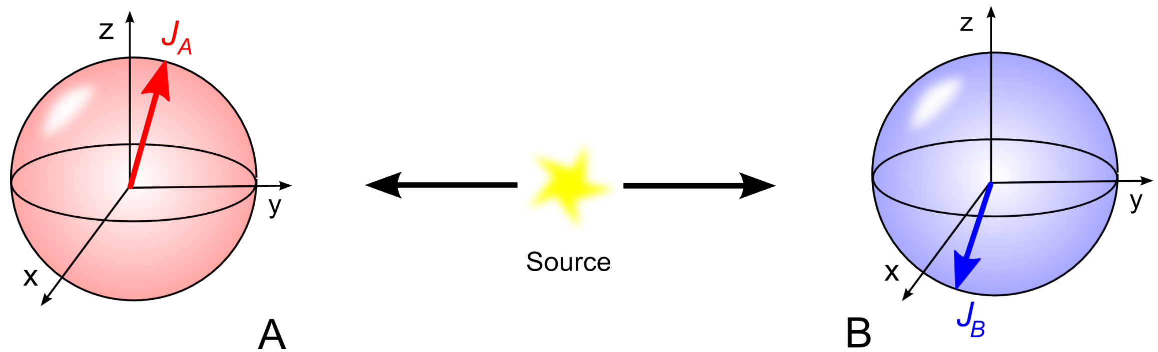
$$E_F = \min_{\{p_R, \psi_R\}} \sum_R \mathrm{P_R}\, E(|\psi_R\rangle\langle\psi_R|)$$

For two qubit systems, this can be worked out, and expressed as the "concurrence" - Wootters. For higher dimensions the question of an entanglement measure is a difficult one.

Concurrence is powerful for two qubits….*necessary and sufficient*
ie will detect entanglement and separability
But these are small systems…
And we know already the Peres positive partial transpose PPT is useful

Technique … to develop *sufficient* conditions for entanglement
based on uncertainty relations – EPR style ….squeezing

Separable Quantum States



$$\rho = \sum_R P_R \rho_A^R \rho_B^R$$

P_R probability

ρ density operator

•Separable states are mixtures of factorised states
"unentangled" states

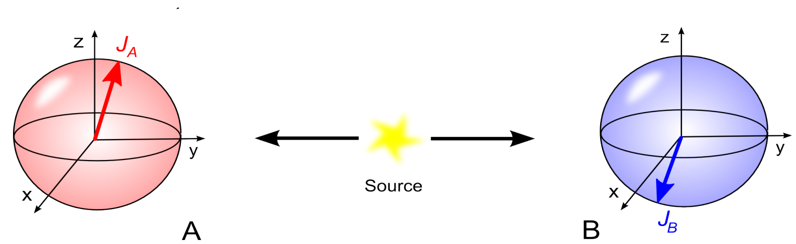
•Local density operators incorporate uncertainty principle
$\Longrightarrow$ *local fuzziness*

• Reduces correlations between A and B - *can't get EPR*

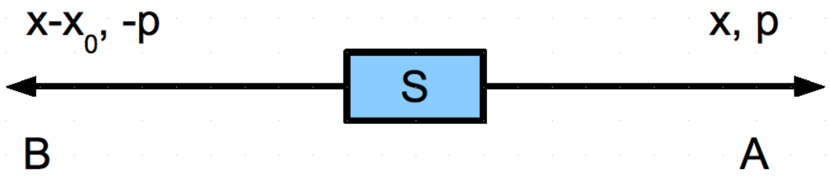
EPR entanglement and squeezing

Entangled states are non-separable: 2 classic examples

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$



$$\delta(x_A - x_B)\delta(p_A + p_B)$$



•Entangled states - greater correlation than separable states for *both conjugate* (non-commuting) observables

•*Variances* of **SUMS OF MOMENTA** and **DIFFERENCES OF POSITION** are zero ie they are squeezed

•How to detect such “EPR” entanglement?.....use squeezing!

Detecting entanglement using squeezing: local uncertainty relations (LUR)

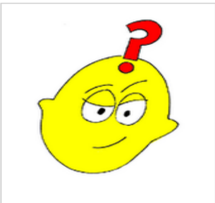
Local Uncertainty Relation Criterion 1:

Assume separability, that the system can be described as a mixture of factorizable states, so that

$$\rho = \sum_R P_R \rho_R^A \rho_R^B ...$$

where P_R is a probability ($\sum_R P_R = 1$) and ρ_R^A is a quantum density operator for a state at site A, and ρ_R^B one for site B, etc. We follow approach of Duan et al (Physical Review Letters (PRL), 2000) and Hofman and Takeuchi (see below for reference) to derive criteria following from this assumption, that are then criteria *sufficient* (but not necessary) to demonstrate entanglement.

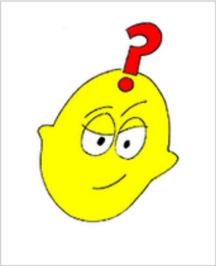
Assuming separability, we can write that the variance of a mixture must not be less than the average of the variances of its components. So if separability holds (no entanglement), we must have



Actually, that result depends only the ρ being a mixture

Detecting entanglement using squeezing: local uncertainty relations (LUR)

$$
\begin{aligned}
\Delta^2(X_A - X_B) + \Delta^2(P_A + P_B) \quad &\geq \quad \sum_R P_R(\Delta_R^2(X_A - X_B) + \Delta_R^2(P_A + P_B)) \\
&= \quad \sum_R P_R[\langle X_A^2\rangle_R + \langle X_B^2\rangle_R - 2\langle X_A\rangle_R\langle X_B\rangle_R] \\
&\quad + \sum_R P_R[\langle P_A^2\rangle_R + \langle P_B^2\rangle_R + 2\langle P_A\rangle_R\langle P_B\rangle_R] \\
&\quad - \sum_R P_R\langle X_A - X_B\rangle_R^2 - \sum_R P_R\langle P_A + P_B\rangle_R^2 \\
&= \quad \sum_R P_R(\Delta_R^2 X_A + \Delta_R^2 P_A + \Delta_R^2 X_B + \Delta_R^2 P_B).
\end{aligned}
$$



Use separability here, because we factorise the A and B moments

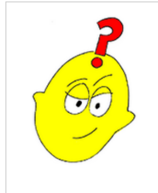
There will be a cancelation of those terms here

Now, each R is a quantum state
This means the uncertainty principle holds for each state R at A and B

Detecting entanglement using squeezing: local uncertainty relations (LUR)

Now for a quantum state, the following uncertainty relation follows from $\Delta X \Delta P \geq 1$. So the "local uncertainty relation" is

$$\Delta^2 X + \Delta^2 P \geq 2$$



ie Separability implies
At least this amount of noise!

The inequality then becomes

$$\Delta^2(X_A - X_B) + \Delta^2(P_A + P_B) \geq 4$$

which gives

$$\Delta^2(X_A - X_B) + \Delta^2(P_A + P_B) < 4$$

as a sufficient criterion for entanglement (note the different forms that appear in the literature depending on the choice of normalisation of the definition of X and P).

Note that quantum mechanics allows the Left hand side to be zero, because the commutator of $X_A - X_B$ and $P_A + P_B$ is zero!

But how do we generate and measure such entanglement?

Remember: lecture 2

How is this squeezing measured?

Combine with large coherent field (laser) using a beam splitter (50/50 mirror) to get a measure of this fluctuation eg

$$
\begin{aligned}
a_{out,+} &= [a_+ + a_-]/\sqrt{2} \\
a_{out,-} &= [-a_+ + a_-]/\sqrt{2}
\end{aligned}
$$

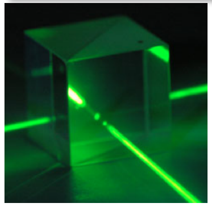
but if a_+ is very large, it can be classical amplitude $Ee^{-i\theta}$- then the photon number difference between the two arms of the beam splitter is

$a^\dagger_{out,+}a_{out,+} - a^\dagger_{in+}a_{in,+} = E(a^\dagger_- e^{i\theta} + a_- e^{-i\theta})$... this becomes X or P depending on the choice of phase θ.

Need to identify the "quantum limit": defined as that for a coherent state, best to take vacuum $|0\rangle$: so measure noise levels with a_- a vacuum, then compare with noise levels when a_- is the squeezed light source.

Measurement of the quadratures X,P

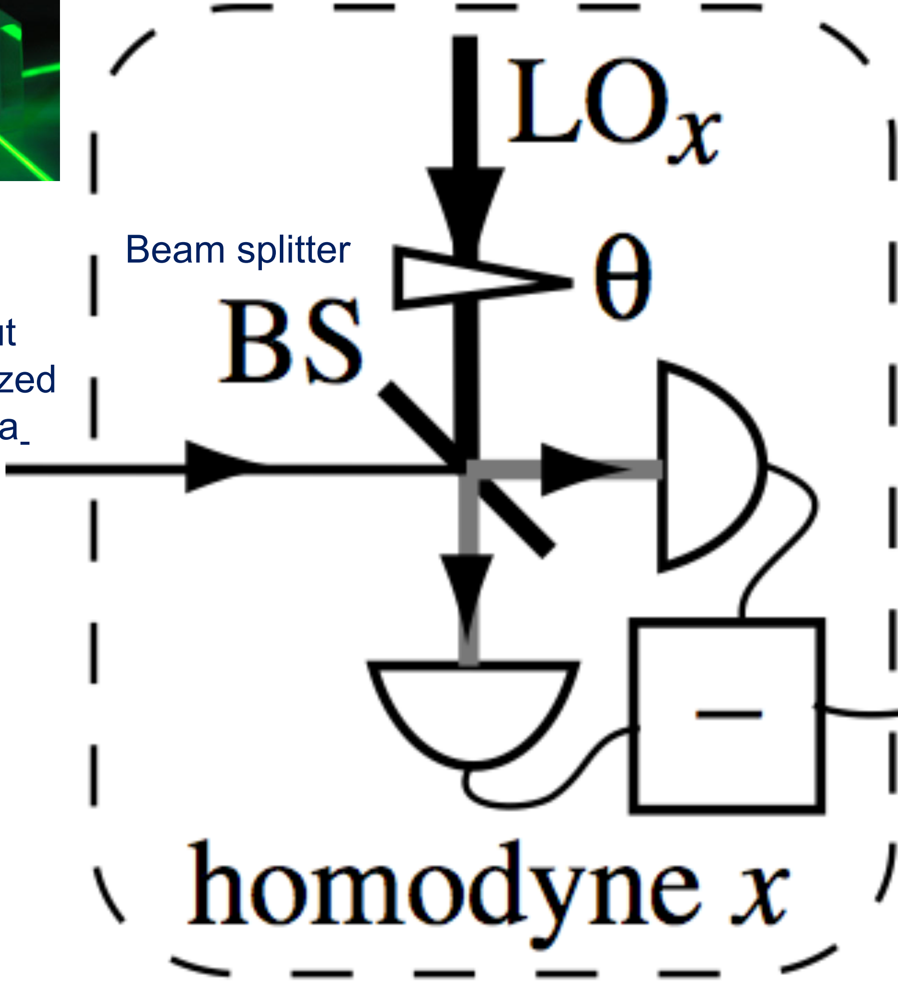
Remember: lecture 2



Strong laser field E a_+:
local oscillator

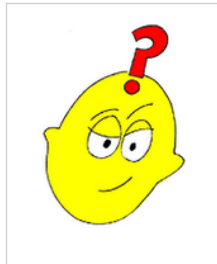
Beam splitter

Input
squeezed
field a_-



Phase difference that selects X or P

Photon number difference measurement - corresponds to EX_θ



Question: what if the second input port a_- has a vacuum state |0> input?
How does variance of number difference vary with θ?

See notes on this

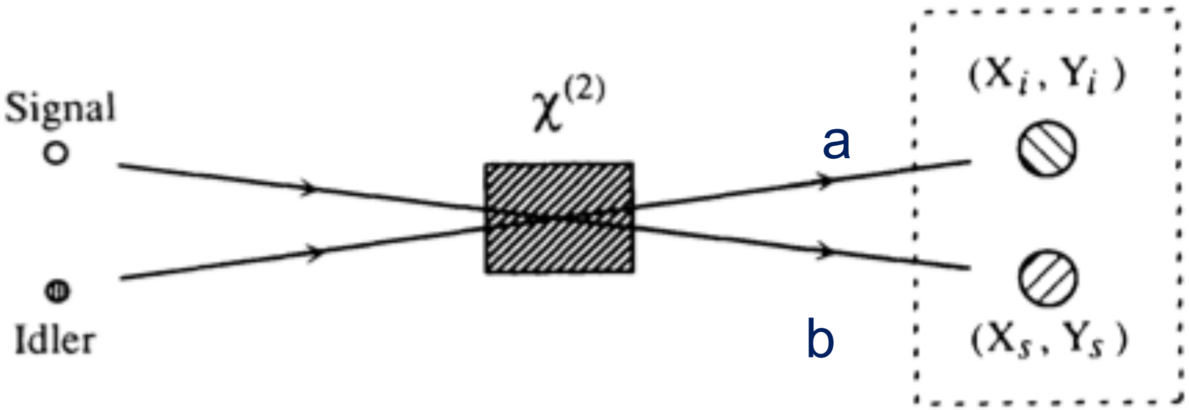
EPR entanglement generated from twin photon sources

Recall lecture 2

BER 25 PHYSICAL REVIEW LETTERS 22 JU

Realization of the Einstein-Podolsky-Rosen Paradox for Continuous Variables

Z. Y. Ou, S. F. Pereira, H. J. Kimble, and K. C. Peng[(a)]
Norman Bridge Laboratory of Physics 12-33, California Institute of Technology, Pasadena, California 91125
(Received 20 February 1992)



E is a pump field

$$H = \kappa E(a^+b^+ + ab)$$

EPR correlated quadrature amplitudes generated using a **two-mode OPO Hamiltonian**
X^A correlated with X^B
P^A anticorrelated with P^B

Note: they use Y to mean P

EPR entanglement using squeezing

$$H = \kappa E(a^{+}b^{+} + ab)$$

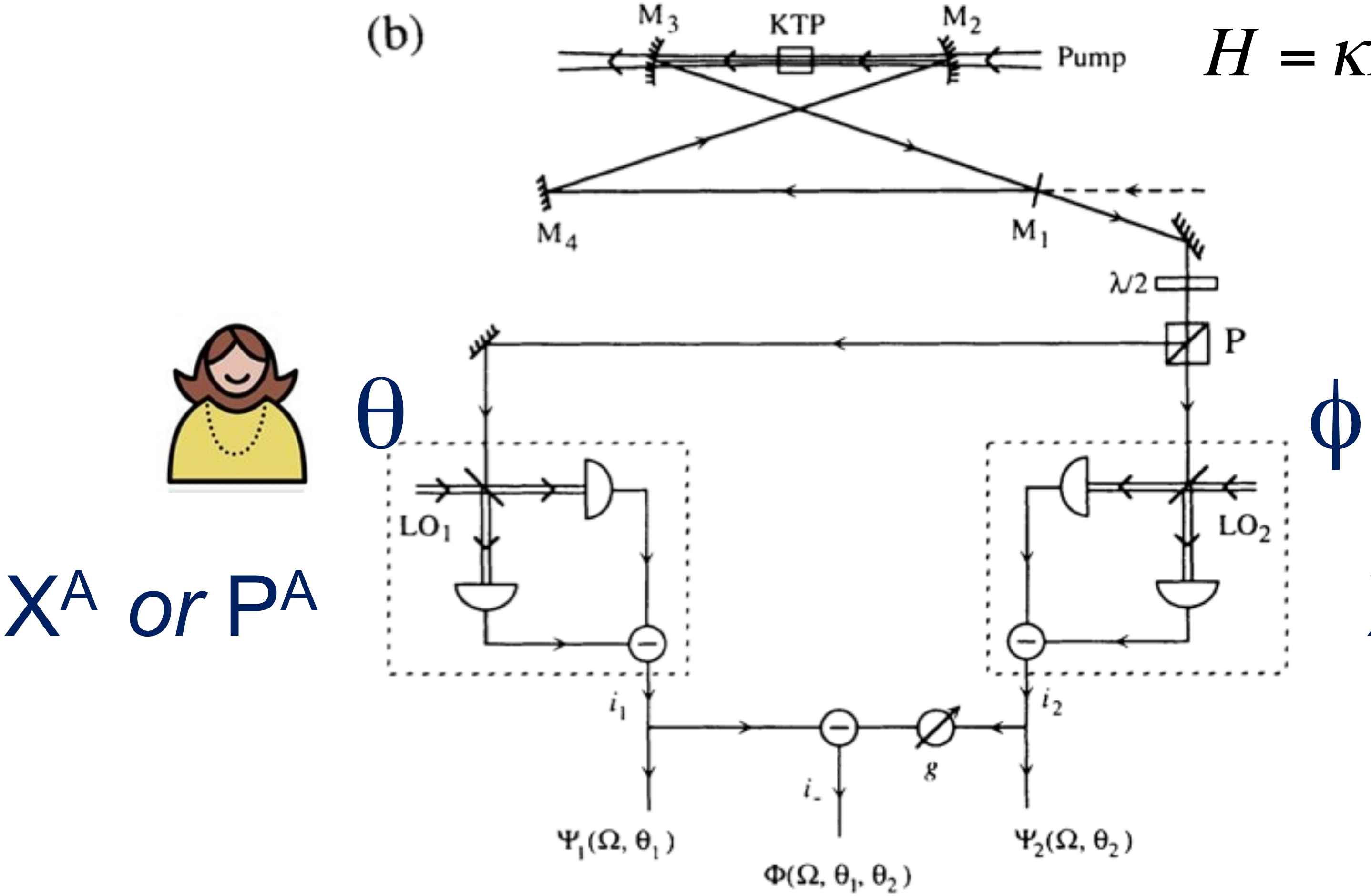
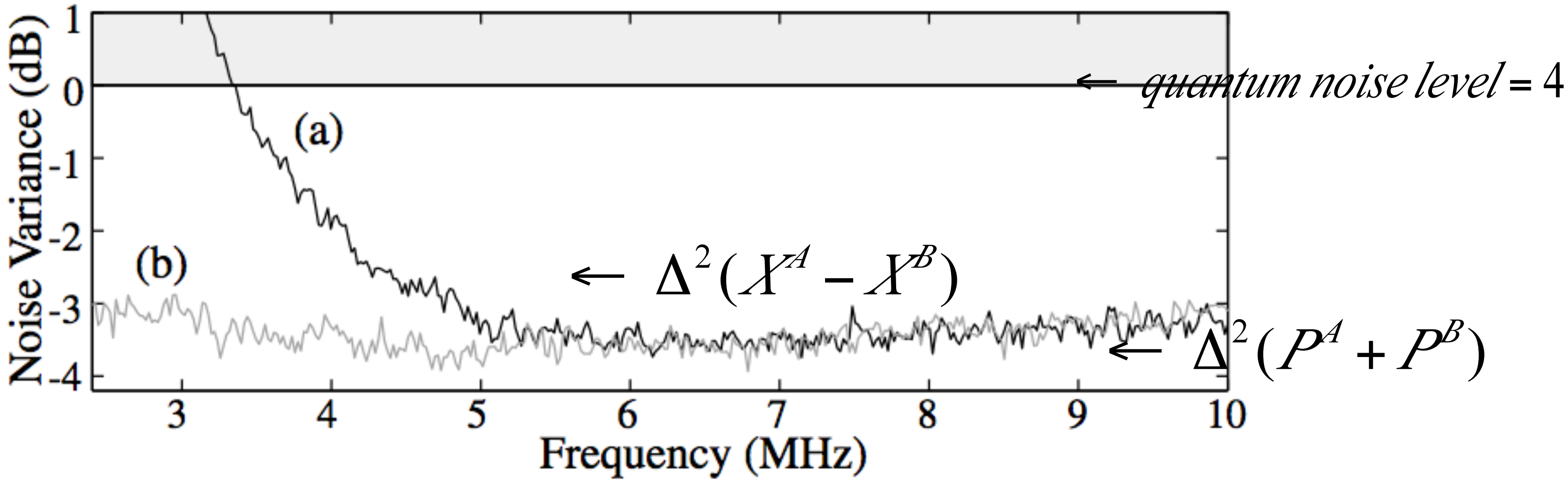


FIG. 1. (a) Scheme for realization of the EPR paradox by nondegenerate parametric amplification, with the optical amplitudes (X_s, Y_s) inferred in turn from (X_i, Y_i). (b) Principal components of the experiment.

EPR entanglement using squeezing

Noise for sum of $X^A - X^B$ ***AND*** $P^A + P^B$ reduced below the quantum limit of 4

$$\Delta^2(X^A - X^B) + \Delta^2(P^A + P^B) < 4$$



Confirms entanglement between A and B

EPR entanglement atomic homodyne

LETTER

doi:10.1038/nature10654

Atomic homodyne detection of continuous-variable entangled twin-atom states

C. Gross[1], H. Strobel[1], E. Nicklas[1], T. Zibold[1], N. Bar-Gill[2]†, G. Kurizki[2] & M. K. Oberthaler[1]

$$H = \kappa E(a^{+}b^{+} + ab)$$

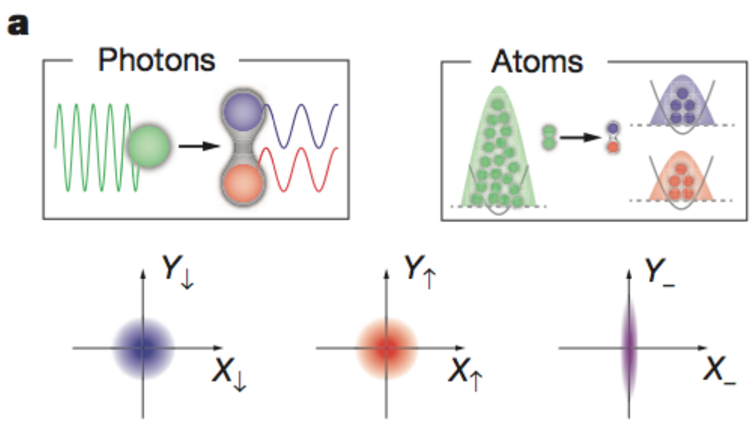


Figure 1 | Analogy to optics and measured population correlations of twin-atom states. a, Parametric down-conversion with light, and the analogy to atomic spin-changing collisions. In a nonlinear medium, effective interactions (top) result in the creation of photon pairs in the signal (red) and idler (blue) modes from pump mode photons (green). The quadratures of the output modes (bottom) are centred around the origin—the individual ones are isotropic, while the two-mode quadratures are squeezed reflecting their correlations (purple). See text for nomenclature. Pair creation due to spin-changing collisions in tight traps is an analogous process in quantum atom optics.

$$\Delta^2(X^A - X^B) + \Delta^2(P^A + P^B) < 4$$

EPR entanglement atomic homodyne

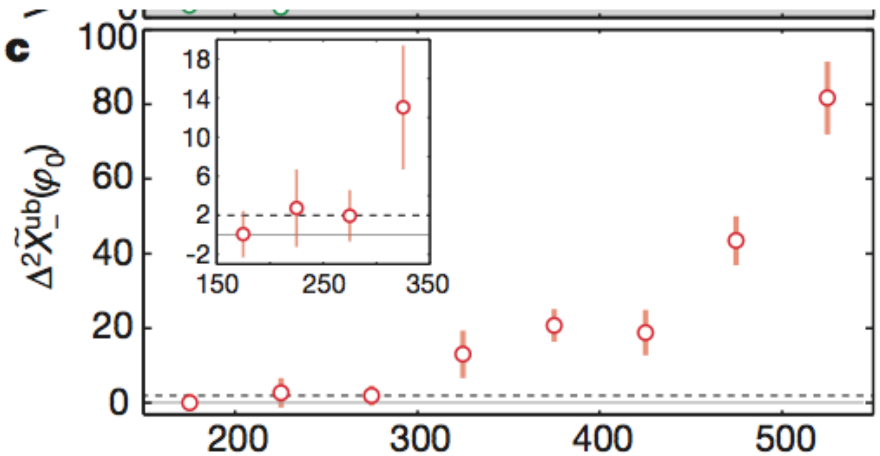
LETTER

doi:10.1038/nature10654

Atomic homodyne detection of continuous-variable entangled twin-atom states

C. Gross[1], H. Strobel[1], E. Nicklas[1], T. Zibold[1], N. Bar-Gill[2]†, G. Kurizki[2] & M. K. Oberthaler[1]

$$\Delta^2(X^A - X^B) + \Delta^2(P^A + P^B) < 4$$



Nearly there

Atoms? Entanglement using spin squeezing

We apply same proof, but apply to spin observables and a spin uncertainty relation

Spin criterion 1:

Another uncertainty relation for fixed spin j is (Hofman and Takeuchi, Physical Review A,68, 032103 (2003))

$$(\Delta J_x^k)^2 + (\Delta J_y^k)^2 + (\Delta J_z^k)^2 \geq j$$

and from ths we can derive criteria for entanglement (Hofman and Takeuchi, PRA,68, 032103 (2003)). Consider two systems A and B: Define collective spin observables

$$J_x = J_x^A \pm J_x^B$$

If we have a separable state (no entanglement), then $\rho = \sum_R P_R \rho_R^A \rho_R^B$.

Entanglement using spin squeezing

Now, because the variance of a mixture can never be less than the average variance of its components, and then because for a factorised state $\rho_R^A \rho_R^B$,

$$\begin{aligned}\Delta(J_x^A \pm J_x^B) &= \langle (J_x^A \pm J_x^B)^2 \rangle - \langle (J_x^A \pm J_x^B) \rangle^2 \\ &= (\Delta J_x^A)^2 + (\Delta J_x^B)^2\end{aligned}$$

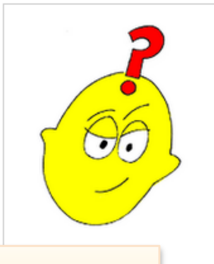
and after using the Local Uncertainty Relation (LUR) we find that separability implies

$$\begin{aligned}(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq \sum_R P_R \{(\Delta J_x)_R^2 + (\Delta J_y)_R^2 + (\Delta J_z)_R^2 \\ &= \sum_R P_R \{(\Delta J_x^A)_R^2 + (\Delta J_y^A)_R^2 + (\Delta J_z^A)_R^2 \\ &\quad + \{(\Delta J_x^B)_R^2 + (\Delta J_y^B)_R^2 + (\Delta J_z^B)_R^2 \\ &\geq 2j\end{aligned}$$

Thus, if

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 < 2j$$

then the two systems A and B are entangled.



Uncertainty relation for two spins C_J

A second uncertainty relation exists for just two spins
We can use this relation to derive an entanglement criterion
Involving the spin measurements X and Z

$$(\Delta J_X^A)^2 + (\Delta J_Z^A)^2 \geq C_J$$

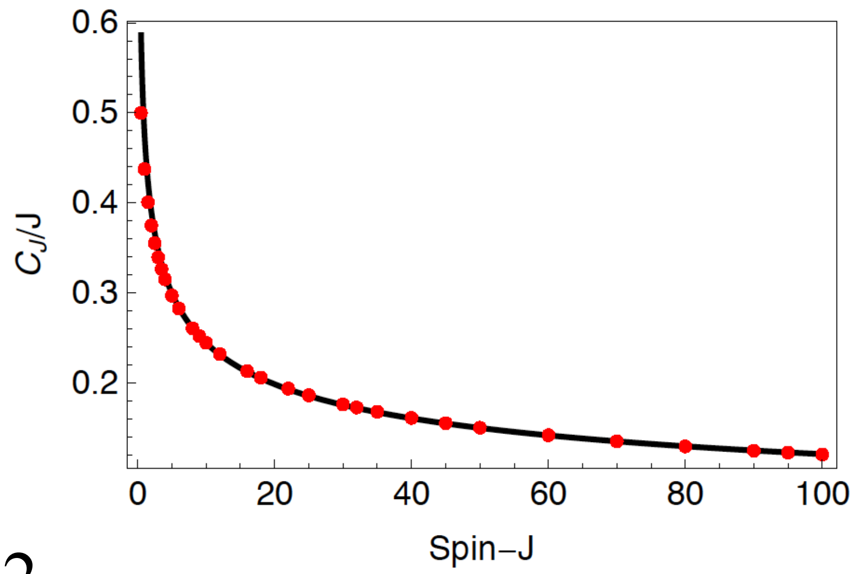
$$(\Delta J_X^B)^2 + (\Delta J_Z^B)^2 \geq C_J$$

For j=1/2

$$(\Delta J_X)^2 + (\Delta J_Z)^2 \geq 1/2$$

Or if we use Pauli spins

$$(\Delta \sigma_X)^2 + (\Delta \sigma_Z)^2 \geq 2$$



Thus, two systems are entangled
if these inequalities are violated

Entanglement using spin squeezing

Exercise: consider the Bell singlet state

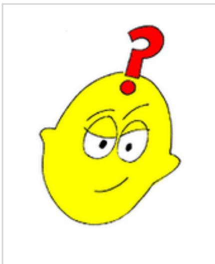
$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B]$$

Will this criterion pick up the entanglement of the Bell state? what is the sum of the variances? Remember the Bohm EPR paradox, for which spins were correlated.

Answer: recall your answer to exercise 1... the EPR paradox. The spins are perfectly correlated in all directions, so the sum of the variances (with appropriate choice of a "sum" or "difference" depending on whether there is the correlation or anticorrelation) is 0.

Exercise: Does this criterion pick up the entanglement of the Werner state mixture? (see article in PRA by Hofman and Takeuchi).

See answer in notes on line



Entanglement using spin squeezing: more!

$$\Delta J_x \Delta J_y \geq |\langle J_z \rangle|/2$$

One can use the product uncertainty relation in a similar way: entanglement is detected if (Giovannetti et al. PRA67, 022320 (2003)).

$$(\Delta J_x)(\Delta J_y) < [|\langle J_z^A \rangle| + |\langle J_z^B \rangle|]/2$$

ALSO a criterion sufficient for entanglement is

$$\Delta^2(J_z^A \mp J_z^B) + \Delta^2(J_y^A \pm J_y^B) < |\langle J_z^A \rangle| + |\langle J_z^B \rangle|$$

So! There are many different LUR criteria for entanglement, using different observables and different uncertainty relations.

These different criteria are useful in different situations

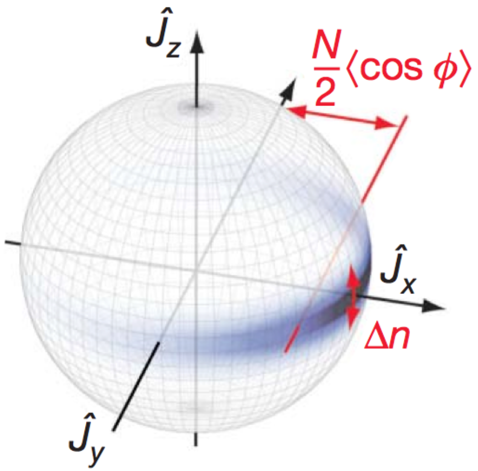
Spin squeezing: useful for detecting entanglement with atoms

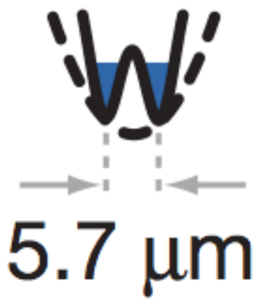
Spin squeezing:

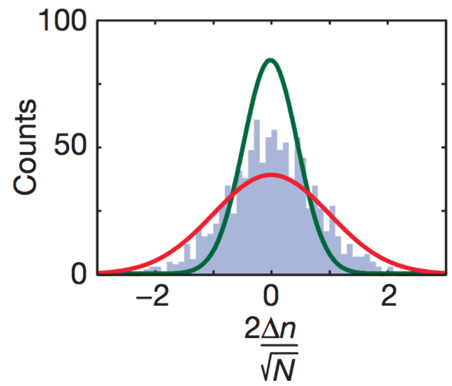
Define spins J_x, J_y, J_z:

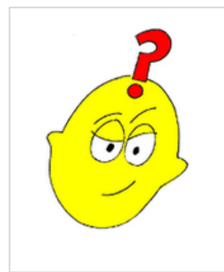
$$\Delta J_x \Delta J_y \geq |\langle J_z \rangle|/2$$

Spin squeezing when $\Delta J_x < \sqrt{|\langle J_z \rangle|/2}$. Spin squeezing has been measured using optical Schwinger spin (polarisation modes) and more recently for cold atoms. Often, the convention is to take $(\Delta J_z)^2 < |\langle J_x \rangle|/2$, and J_Z is measured as the Schwinger number difference: $J_z = (a_1^\dagger a_1 - a_2^\dagger a_2)/2$, so squeezing shows as a reduced number difference fluctuation (ie is phase insensitive).









Some spin squeezing formalism

Two-level atom/ spin formalism

one level of an atom is denoted $|1\rangle$, the second level $|2\rangle$

define spin operators according to:

$\sigma = |0\rangle\langle 1|$, $\sigma^\dagger = |1\rangle\langle 0|$, $\sigma_z = (|1\rangle\langle 1| - |0\rangle\langle 0|)/2$

If we have a large number N of such atoms: (levels); or two polarisation modes $\pm$ that can be occupied by large number of photons, or two levels that can be occupied by a large number of particles then it extremely useful to

define Schwinger spins (check commutation relations using boson relations: $([a^\dagger, a] = 1)$)...use this to check my relations!)

$$
\begin{aligned}
J_z &= (a_1^\dagger a_1 - a_2^\dagger a_2)/2 \\
J_x &= (a_1^\dagger a_2 + a_2^\dagger a_1)/2 \\
J_y &= (a_1^\dagger a_2 - a_2^\dagger a_1)/2i
\end{aligned}
$$

so eg $a_1^\dagger a_1$ is the number of particles occupying level (or "state") 1; and similarly $a_2^\dagger a_2$ is number occupying level 2. Thus J_z gives "number difference" between two levels. If we have a fixed number N of particles, then $a_1^\dagger a_1 + a_2^\dagger a_2$ is conserved as the total number N, which means $j = N/2$ is the "spin" of the system.

Spin squeezing: will be useful for detecting entanglement with atoms

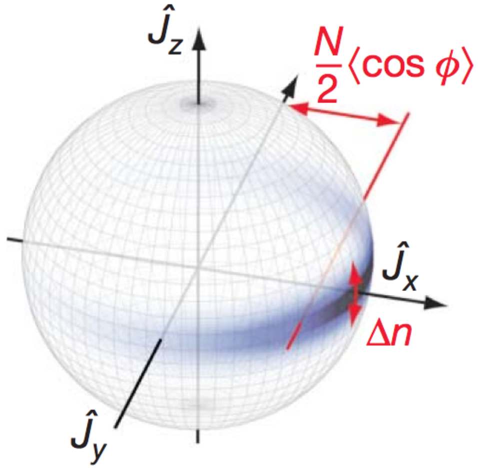
Spin squeezing:

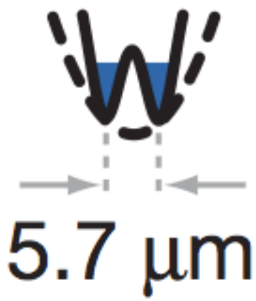
See notes on line for how the spin squeezed states can be generated

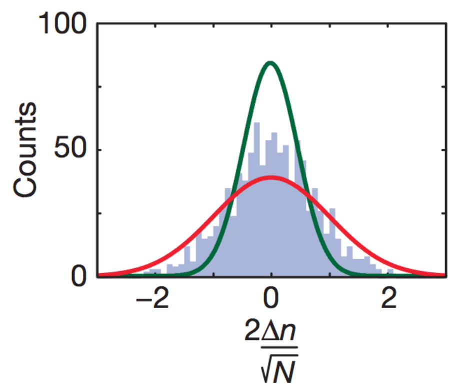
Define spins J_x, J_y, J_z:

$$\Delta J_x \Delta J_y \geq |\langle J_z \rangle|/2$$

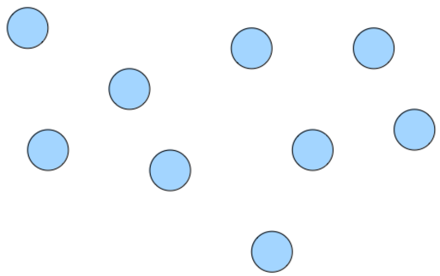
Spin squeezing when $\Delta J_x < \sqrt{|\langle J_z \rangle|/2}$. Spin squeezing has been measured using optical Schwinger spin (polarisation modes) and more recently for cold atoms. Often, the convention is to take $(\Delta J_z)^2 < |\langle J_x \rangle|/2$, and J_Z is measured as the Schwinger number difference: $J_z = (a_1^\dagger a_1 - a_2^\dagger a_2)/2$, so squeezing shows as a reduced number difference fluctuation (ie is phase insensitive).







Entangled atoms: spin squeezing criterion



- N identical spin ½ systems (atoms) denoted by i
- Define and measure *collective* spins J_X, J_Y, J_Z.

$$J_Z = \sum_{i=1}^{N} J_Z^i$$

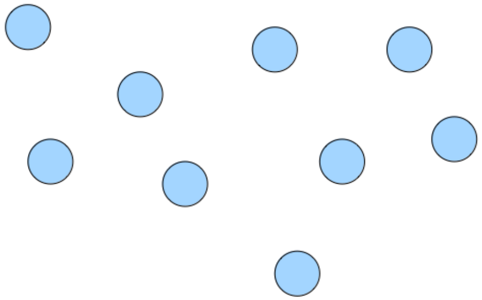
(not able to test EPR/ nonlocality)

- Each system satisfies Local Uncertainty Relation

$$\Delta J_Z^i \Delta J_Y^i \geq \left| \left\langle J_X^i \right\rangle \right| / 2$$

- What if the atoms are not entangled?

Entangled atoms: spin squeezing criterion



$$J_Z = \sum_{i=1}^{N} J_Z^i \qquad \Delta J_Z^i \Delta J_Y^i \geq \left|\left\langle J_X^i \right\rangle\right| / 2$$

If there is no entanglement, we can write the density operator as a mixture of product states

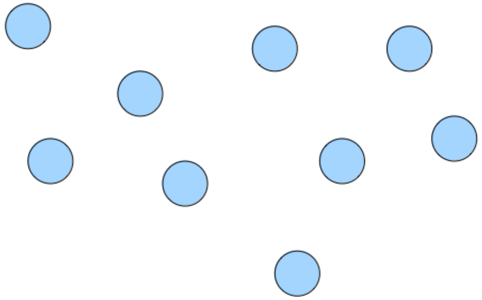
$$\rho = \sum_R P_R \rho_1^R \ldots\ldots \rho_N^R$$

This will constrain the statistics of the system

(1) First we note that for a single spin j system (finite dimensionality), there is a constraint on how large the variance in spin can get!

$$\Delta J_Y^i \leq j \qquad \Rightarrow \qquad \Delta^2 J_Y^i \leq 1/4 \quad where \quad j = 1/2$$

Entangled atoms: spin squeezing criterion



$$J_Z = \sum_{i=1}^{N} J_Z^i$$

Assuming NO entanglement, we write

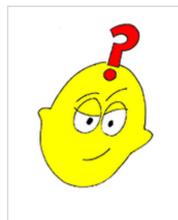
$$\rho = \sum_R P_R \rho_1^R \ldots\ldots \rho_N^R$$

Now consider the variance in the collective spin.
(2) We use convexity and separability to get the first step (compare with previous proof)

$$\Delta^2 J_Z \geq \sum_R P_R \sum_{i=1}^{N} (\Delta_R^2 J_Z^i) \geq \sum_R P_R \sum_{i=1}^{N} (\Delta_R^2 J_Z^i)_{\min}$$

(3) What is the minimum variance for any single spin system?

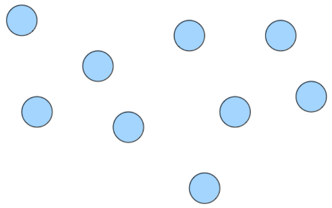
Use $\quad \Delta J_Z^i \Delta J_Y^i \geq \left|\left\langle J_X^i \right\rangle\right| / 2 \quad$ we find



$$\Delta^2 J_Z^i \geq \left|\left\langle J_X^i \right\rangle\right|^2 / (2j)^2 \quad \rightarrow \Delta^2 J_Z^i \geq \left|\left\langle J_X^i \right\rangle\right|^2 \quad \textit{where } j = 1/2$$

Applies to any R

Entangled atoms: spin squeezing criterion

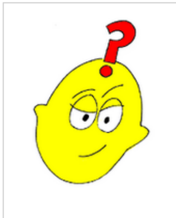


Now use Cauchy Schwarz inequality

$$\Delta^2 J_Z \geq \sum_R P_R \sum_{i=1}^{N} (\Delta_R^2 J_Z^i) \geq \sum_R P_R \sum_{i=1}^{N} (\Delta_R^2 J_Z^i)_{\min}$$

$$\geq \sum_R P_R \sum_{i=1}^{N} \left\langle J^i{}_X \right\rangle^2$$

$$\geq \left| \left\langle J_X \right\rangle \right|^2 / N$$



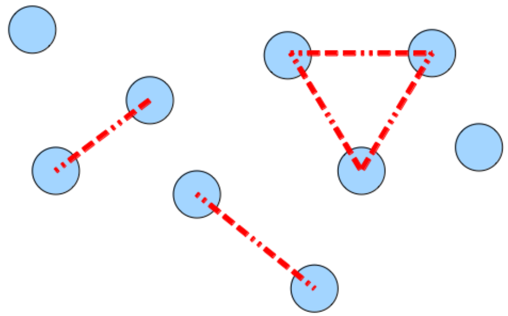
The Cauchy Schwarz applications are nontrivial - see extra notes on line

Violation of this "spin squeezing inequality" indicate entanglement!

If we observe a collective spin squeezing good enough so $\Delta^2 J_X < \left| \left\langle J_Z \right\rangle \right|^2 / N$

Then we have entanglement (between at least 2 systems)

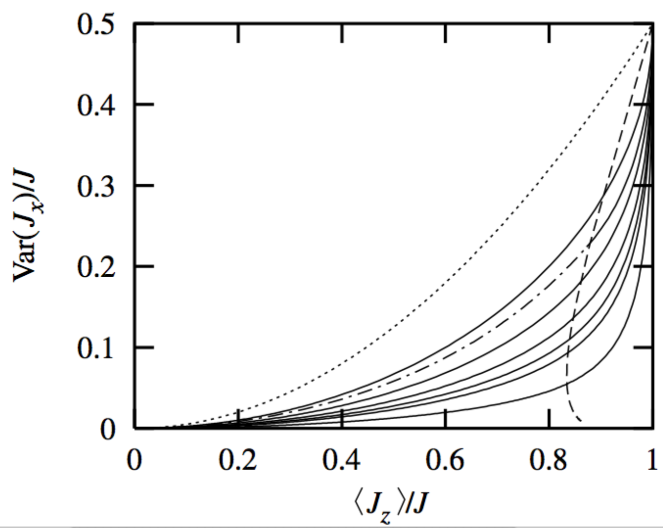
But how many atoms entangled?



Extent of spin squeezing determines degree of entanglement

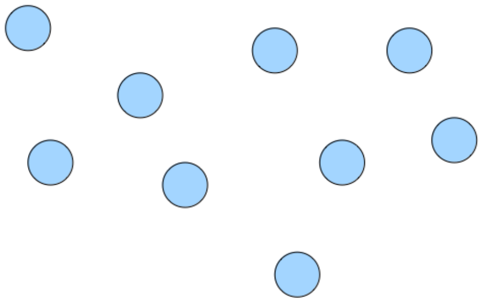
$$\rho = \sum_R P_R \rho_1 .....\rho_{N'}$$

$$\Delta^2 J_Z \geq \sum_R P_R \sum_{i=1}^{N} (\Delta_R^2 J_Z)_i \geq \sum_R P_R \sum_{i=1}^{N} (\Delta_R^2 J_Z)_{i,\min}$$

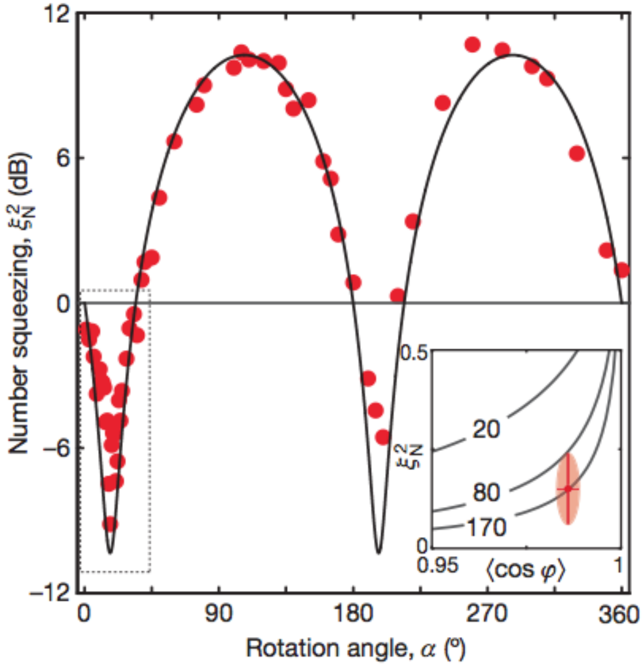


As J increases, we can get better spin squeezing
thus, the amount of spin squeezing gives information about the minimum value of j
Which gives information about the minimum number of atoms are entangled

Entangled atoms: spin squeezing criterion

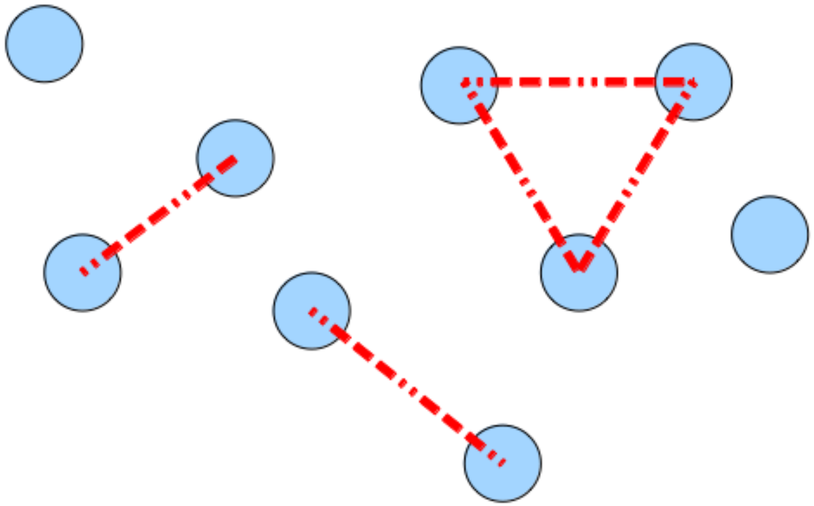


$$(\Delta J_Z)^2 < \left|\langle J_X \rangle\right|^2 / N$$



~100 atoms entangled

Genuine multipartite entanglement/ nonlocality



Are 2 or 3 particles really entangled?

$$|\Psi\rangle_{GHZ} = \frac{1}{\sqrt{2}}\left(|111\rangle - |000\rangle\right)$$

$$|\Psi\rangle_{W states} = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right)$$

How to detect it?

Svetlichny asked same questions of nonlocality 1987

Svetlichny's Bell inequality

PHYSICAL REVIEW D VOLUME 35, NUMBER 10 15 MAY 1987

Distinguishing three-body from two-body nonseparability by a Bell-type inequality

George Svetlichny
Departamento de Matematica, Pontificia Universidade Catolica, Rio de Janeiro, Rio de Janeiro, Brazil
(Received 24 April 1986)

We derive an inequality, violated by quantum mechanics, that in a three-body system can detect three-body correlations that cannot be reduced to mixtures of two-body ones related locally to the third body.

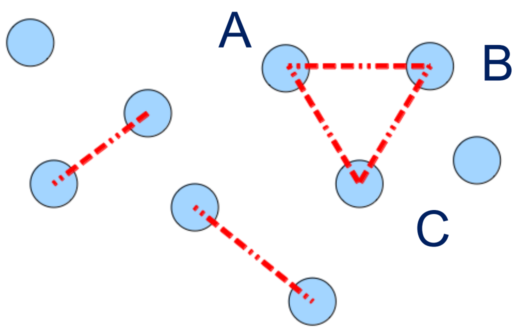
Physicists generally agree that quantum mechanics gives accurate and at times remarkably accurate numerical predictions. The existing body of experimental evidence, however, is not qualitatively diverse enough to warrant the approximations or the phenomenological input that must be used in the theoretical treatment. Nuclear, condensed matter, and the bulk of elementary-partice physics fall into this category.

Abbreviating $E(A_iB_jC_k)$ to $E(ijk)$ our inequalities that follow from the limited entanglement hypothesis are thus

$$|E(111)+E(112)+E(211)-E(212)+E(121)-E(122)-E(221)-E(222)| \leq 4\,, \tag{5}$$

$$|E(111)+E(112)-E(211)-E(212)-E(121)-E(122)-E(221)+E(222)| \leq 4\,. \tag{6}$$

But *how many* sites/ particles are *genuinely* entangled?

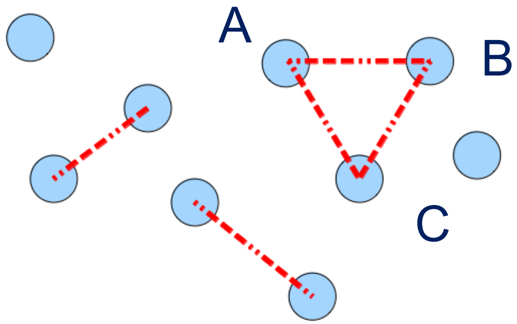


Verifying genuine *tripartite entanglement*:
need to measure entanglement but exclude that 2 party entanglement can completely describe the statistics

To prove bipartite entanglement AB, we exclude

$$\rho = \sum_R P_R \rho_R^A \rho_R^B$$

Criteria for multipartite entanglement? partial factorisation



Verifying genuine tripartite *entanglement*:
need to exclude that 2 party entanglement can describe the statistics

To prove tripartite entanglement, we need to exclude all three forms for ρ
Allowing for 2 party entanglement

$$\rho = \sum P_R \rho_A^R \rho_{BC}^R$$

➔ ***B and C can be** entangled*

$$\rho = \sum P_R \rho_{AB}^R \rho_C^R$$

$$\rho = \sum P_R \rho_B^R \rho_{AC}^R$$

Verifying genuine tripartite Bell *nonlocality*: need to exclude *all* 2 body nonlocality

Quantum cryptography

Cryptography

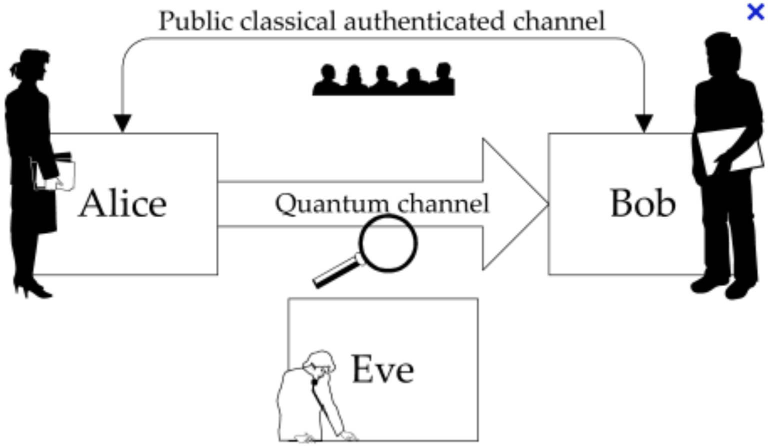
Source prepared in a Bell state

Alice and Bob use correlated bits (up/ down) state for a quantum key *Ekert*

They *randomly* switch the choice of spin measurement …. decision made *locally* after particles are in flight (θ, θ', x,y;ϕ,ϕ',x,y)

Get together later, check measurement angles (only)- where same, the result will be opposite….share a secret key (bit values) QKD

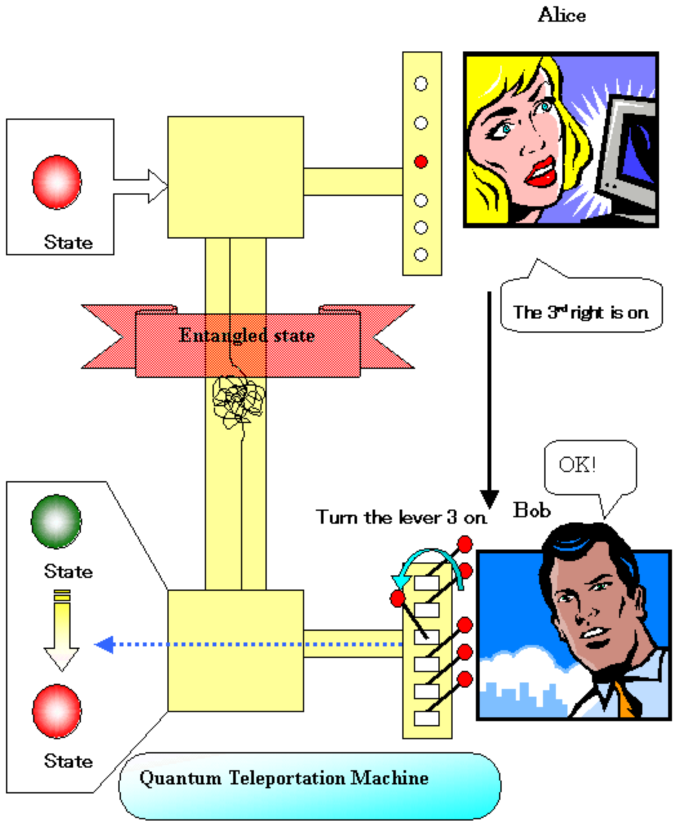
Security? –check results for violation of Bell inequality…..If Eve has eavesdropped, there will be *no violation* of Bell inequality



$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle\right)$$

Quantum teleportation

Transferring a quantum state to a different location



Problems with teleportation

$|\psi_C\rangle$

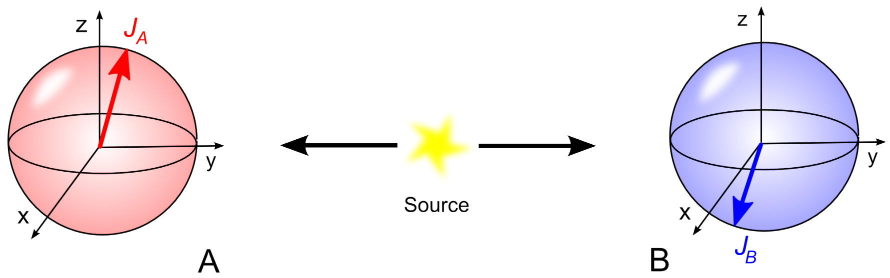
- Alice wants to send a quantum state to Bob

$$|\psi_C\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$$

- She cannot measure the state and then send the results ("measure and preparation")

- If she sends the state itself, it might deteriorate on the way or take too long

Four Bell States



- 4 Bell states (the maximally nonlocal and entangled states)

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle\right)$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle\right)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\uparrow\rangle - |\downarrow\rangle|\downarrow\rangle\right)$$

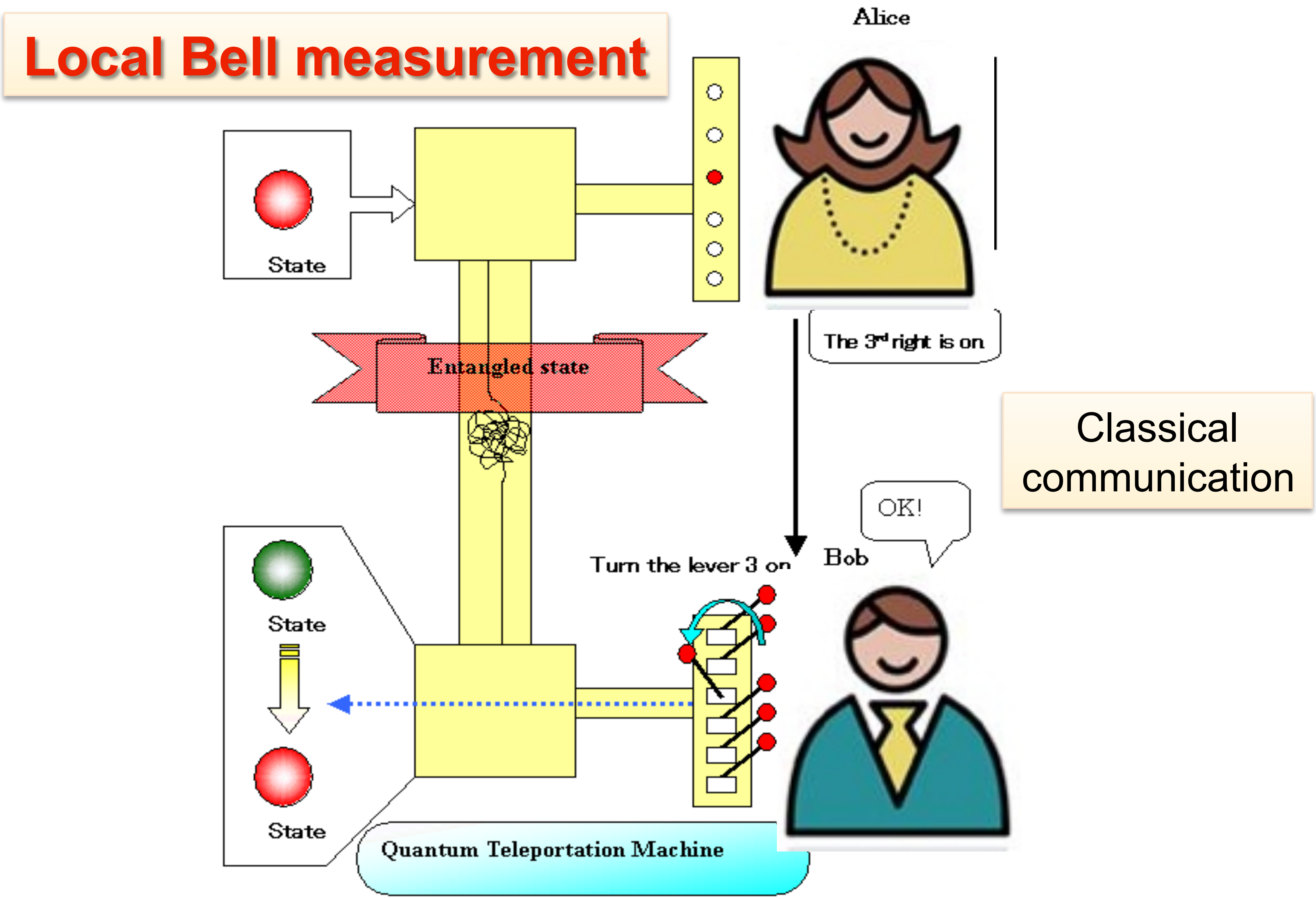
$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle|\uparrow\rangle + |\downarrow\rangle|\downarrow\rangle\right)$$

- Form a basis for all 2 qubit systems

Teleportation protocol

- Alice has a qubit $|\psi_C\rangle$ that she wants to teleport
- Alice and Bob share a **two qubit entangled** Bell state, let's say $|\phi^+_{AB}\rangle$
- She performs a **Bell measurement** on her local states AC, and sends Bob the result.
- Bob performs a transformation of his qubit B, according to Alice's Bell measurement result and qubit B becomes a replica $|\psi_C\rangle$ of qubit C

Quantum teleportation



Calculate?

- Before Alice's "Bell measurement" the state is

$$\left|\phi_{AB}^{+}\right\rangle\left|\psi_{C}\right\rangle$$

- which can be re-expressed in Alice's Bell state basis as

$$\left|\phi_{AC}^{+}\right\rangle(\alpha\left|\uparrow\right\rangle_{B}+\beta\left|\downarrow\right\rangle_{B})$$
$$+\left|\phi_{AC}^{-}\right\rangle(\alpha\left|\uparrow\right\rangle_{B}-\beta\left|\downarrow\right\rangle_{B})$$
$$+\left|\psi_{AC}^{+}\right\rangle(\beta\left|\uparrow\right\rangle_{B}+\alpha\left|\downarrow\right\rangle_{b})$$
$$+\left|\psi_{AC}^{-}\right\rangle(\beta\left|\uparrow\right\rangle_{B}-\alpha\left|\downarrow\right\rangle_{B})$$

- Alice's local **"Bell measurement" gives the result indicating one of the Bell states**

Calculate?

$$\left|\phi_{AC}^{+}\right\rangle(\alpha\left|\uparrow\right\rangle_B+\beta\left|\downarrow\right\rangle_B)$$
$$+\left|\phi_{AC}^{-}\right\rangle(\alpha\left|\uparrow\right\rangle_B-\beta\left|\downarrow\right\rangle_B)$$
$$+\left|\psi_{AC}^{+}\right\rangle(\beta\left|\uparrow\right\rangle_B+\alpha\left|\downarrow\right\rangle_B)$$
$$+\left|\psi_{AC}^{-}\right\rangle(\beta\left|\uparrow\right\rangle_B-\alpha\left|\downarrow\right\rangle_B)$$

- By performing a Bell measurement on AC state Alice knows that Bob's system "collapses" (is REDUCED) into one of the above states
- EG if she measures $\left|\phi_{AC}^{-}\right\rangle$ what is Bob's state?

$$(\alpha\left|\uparrow\right\rangle_B-\beta\left|\downarrow\right\rangle_B)$$

Calculate?

- By sending the result classically to Bob, Alice instructs Bob which transformation to perform on his state – to get his qubit in the form of

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle$$

- Bob operates locally using Pauli matrices – which Pauli matrix does he use if he gets the signal for the Bell state $|\phi^-_{AC}\rangle$? (use σ_Z)

$$\alpha|\uparrow\rangle_B - \beta|\downarrow\rangle_B \rightarrow |\psi\rangle$$

Outline

1. Non-locality and quantum mechanics

- Einstein's (EPR) spooky action at a distance 1935
- Schrodinger's cat 1935
- Bell's theorem 1965
    - Bell and EPR experiments
- GHZ's extreme multiparticle quantum nonlocality

2. Introduce formalism of entanglement

- Density operator – mixed states
- Inseparability of density matrix
- Pauli spin examples
- Werner states
- Peres PPT criterion and concurrence
- Quadrature squeezing and spin squeezing
- CV Variance and spin squeezing criteria for entanglement

3. Applications

- Quantum cryptography and quantum teleportation