Lectures on Entanglement

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July 5, 2012

1 History: Quantum Nonlocality

1.1 Quantum mechanics and Reality

Quantum mechanics (QM) is fundamentally different to classical mechanics. Results of measurements are not given by parameters that can considered predetermined − a superposition of two states - "the system cannot be considered to be in one state or the other until measured". The quantum wavefunction contains inherent uncertainty- we know that, we can accept that - but the issue is more subtle than this. It is as though in QM the measurement "brings about" the result.- OK, we might say, that is OK, there is an interaction between the system and measurement apparatus, that makes sense for small systems anyway, ...

TWO problems here arise

Schrodinger cat: how to interpret the QM prediction of existence of macroscopic superpositions

Einstein's spooky action at a distance: 1935 Most important: Einstein's showed well, ok but there are situations where the measurements can apparently instantaneously "bring about a result" to a *distant* system. Now, how can the issue of interaction between measurement apparatus and system be relevant there?

These issues are are to do with ENTANGLED STATES which are at the core of the difference between QM and classical mechanics. Schrodinger introduced word "entangled" states to the quantum states that seem to give this effect.

1.2 Einstein-Rosen-Podolsky (EPR) paradox

Einstein was unhappy about the assumption that quantum mechanics (QM) may be a complete theory. EPR formulated a powerful argument that quantum mechanics was incomplete in1935. The argument is based on assumptions about the truth of "local realism" (LR), which had so far been taken for granted. In essence, the argument assumes LR, and based on the predicted existence by QM of certain *entangled* states, it logically argues that QM is incomplete. Thus, "completeness of QM" and LR ("locality") are incompatible, at least if you have entanglement.

The EPR argument thus begins a journey of understanding into "quantum nonlocality", and at the heart of this journey is the concept of entanglement.

The EPR paradox: Bohm's example with spins

Look up EPR's original paper in Physical Review A. The original argment was presented in terms of position and momentum, but we will examine Bohm's version of the argument using spins.

Consider two spatially separated particles in the singlet spin 1/2 state

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B]$$

where $|\uparrow\rangle_k$ is the spin "up" eigenstate $|j,m\rangle$ for particle $k = A, B$, where $j = 1/2$, $m = \pm 1/2$. We can consider the three spin measurements J_x, J_y, J_z that can be performed on each particle (we will denote the particle with a superscript).

STEP 1

There is a *perfect* correlation (anti-correlation) between the spins J_z^A and J_z^B of particle A and B.

Exercise 1. Show that there is *also* a perfect correlation between the spin measurements J_x^A and J_x^A, and J_y^B and J_y^B, for this spin singlet state.

We think about correlation between spatially separated systems in the slides, by considering Bell's famous example of Dr Bertlmann socks. This helps us understand what EPR meant when they talked about "reality" and "elements of reality".

STEP 2

Assume there is no "spooky action at a distance" ie. that locality holds. This means that the measurement at one location does not influence the result of the spin measurement at the second location. The measurement events are to be spacelike separated to fully justify this assertion (see slides). EPR introduce premises, as stated in the slides, which these days are often referred to as "local realism."

STEP 3

If Alice performs a spin J_x measurement on her particle, she can predict what Bob will get if he performs a measurement J_x on his particle. LR $\Longrightarrow$ "element of reality" to describe Bob's spin -ie Bob's spin was predetermined according to parameters describing his local state- before she performs the measurement. After all, Alice cannot "steer" Bob's state "at a distance" into one of definite spin. His spin was predetermined and since she can predict it definitely, is predetermined to be a definite value $\lambda_x = +1/2$ OR $-1/2$.

But the same is true of spin J_y ie Bob's system has associated with it predetermined *definite* spins for x, y and z ie there exists three hidden variables for Bob's measurements $\lambda_x = \pm 1/2$, $\lambda_y = \pm 1/2$ and $\lambda_z = \pm 1/2$, all of which are either $+1/2$ or $-1/2$.

STEP 4

This is just describing classical correlation- nothing new here! ....except when you consider the quantum uncertainty relation/ commutation rules for spin: which states there is NO quantum state with three simultaneously (predetermined) *definite* spins.

STEP 5

EPR argued, based on assumed validity of local realism, that QM is incomplete!

1.3 Schrodinger's reply- entangled states and Schrodinger's cat

Schrodinger responded to EPR's paper, by way of several essays. The most famous introduces the paradox of Schrodinger's cat.

The first stage is a microscopic system in a superposition state $\{|\uparrow\rangle + |\downarrow\rangle\}/\sqrt{2}$, which might be a spin 1/2 particle is a superposition of "spin up" or "spin down" travelling toward a Stern-gerlach apparatus, (OR it might be an alpha particle (that has decayed with 50% chance).

The next stage is a measurement device coupled to the spin eg Stern Gerlach apparatus (or Geiger counter). The coupling takes place, as an interaction H_{int}, so that if $|\uparrow\rangle$ is the initial state and that of the detector is $|i\rangle$ then the final state of the detector and system is $|\uparrow\rangle_{needle}|\uparrow\rangle$ which means needle pointing "up". Similarly, if the initial state is $|\downarrow\rangle$, the final state of detector and system is $|\downarrow\rangle_{needle}|\downarrow\rangle$ which means needle pointing down. Now because of the linearity of Schrodinger's equation, if the system is initially in the superposition $\{|\uparrow\rangle + |\downarrow\rangle\}/\sqrt{2}$, then the final state of the detector is

$$\{|\uparrow\rangle_{needle}|\uparrow\rangle + |\downarrow\rangle_{needle}|\downarrow\rangle\}/\sqrt{2}$$

which is an example of an *entangled* state. Schrodinger considered that the pointer needle is coupled to another system, a trigger to release poison if needle is pointing up. This poison kills cat that is located in a box, with the microscopic system. Considering the same argument as above, the final state of the system will be

$$\{|dead\rangle_{cat}|\uparrow\rangle + |alive\rangle_{cat}|\downarrow\rangle\}/\sqrt{2}$$

How do we understand the meaning of this result, in which the cat is in a superposition of alive and dead? The whole system is in a box, and an observer can look in to measure the stae of the cat. But when did the cat actually die? Was it alive OR dead before the observer peered in the box? In other words, at some stage the state becomes a mixture of alive $|dead\rangle$ and $|alive\rangle$, and this process is called state reduction. The ususal interpretation is that coupling to the environment (dissipation into a large reservoir) will cause "decay" or "decoherence" of the superposition, so it becomes a mixture. But to many this is an unsatisfactory explanation, since quantum mechanics is assumed to apply to all systems. Decoherence environments exist in principle, what happens then? Alternative theories of eg Diosi, Penrose, propose additions to quantum mechanics, that will cause a state reduction for massive objects in superposition states.

1.4 Bell's Theorem

The EPR argument was debated for some years, but the next real development came with Bell's Theorem in 1965-66. Bell's work was important because it gave a way to directly compare the predictions of LR (via all local hidden variable theories) with the predictions of QM. He showed they were incompatible: thus QM or LR is wrong! EPR wouldn't have thought this!

Bell was attempting to construct a theory that would include the hidden parameters λ ie the predetermined spins, and still be consistent with EPR's no spooky action at a distance premise. He couldn't. Here is his argument.

STEP 1:

Exercise 2: Work out the QM prediction for Bell's hypothetical experiment.

Go back to the singlet state and consider arbitrary spin directions: define the spin observable

$$J_\theta = \cos\theta J_x + \sin\theta J_y$$

Work out the prediction for the measurable expectation value of the spin product $E(\theta, \phi) \equiv \langle J_\theta^A J_\phi^B \rangle$:

STEP 2:

What do local hidden variable theories (LHV) predict? All LHV theories put constraints on the expectation of the spin product according to *Bell inequalities.*

Consider first the ideal case of the singlet state which gives perfect correlation between the spin results. Recheck your calculation of Ex 1, to see that the anti-correlation is perfect between J_θ^A and J_θ^B *for all* spins θ.

Then suppose EPR are right ie local realism is right, and there exist hidden parameters λ_θ^k to describe the spins for Bob ($k = B$) and for Alice ($k = A$). For simplicity, we can use Pauli spins, so the outcome for "spin" measurement is $+1$ or -1.

Then the value of λ_θ^A and λ_ϕ^B is *always either* $+1$ or -1.

Now consider the following construction for a two-setting experiment: ie two angles at each location

$$B = E(\theta, \phi) - E(\theta', \phi) + E(\theta, \phi') + E(\theta', \phi')$$

Exercise 3:
Constuct a Table of *all* possibilities for the LR (LHV) prediction. If spins predetermined:

$$\begin{aligned} B &= \langle \lambda_\theta \lambda_\phi \rangle - \langle \lambda_{\theta'} \lambda_\phi \rangle + \langle \lambda_\theta \lambda_{\phi'} \rangle + \langle \lambda_{\theta'} \lambda_{\phi'} \rangle \\ &= \langle \lambda_\theta \lambda_\phi - \lambda_{\theta'} \lambda_\phi + \lambda_\theta \lambda_{\phi'} + \lambda_{\theta'} \lambda_{\phi'} \rangle \equiv \langle B_\lambda \rangle \end{aligned}$$

Outcomes for B according to LR:

λ_θ	$\lambda_{\theta'}$	λ_ϕ	$\lambda_{\phi'}$	$Prod(\theta,\phi)$	$Prod(\theta',\phi)$	$Prod(\theta,\phi')$	$Prod(\theta',\phi')$	B_λ
+1	+1	+1	+1	+1	+1	+1	+1	2
+1	+1	+1	-1	+1	+1	-1	-1	-2

In fact, we arrive at the BELL INEQUALITY (Clauser-Horne-Shimony-Holt CHSH)

$$|B| = |E(\theta, \phi) - E(\theta', \phi) + E(\theta, \phi') + E(\theta', \phi')| \le 2$$

This proof can be generalised to cases that are *not* perfectly correlated, so that there is a hidden variable state $\{\lambda\}$ for each location that gives an average prediction for the spin. Such a theory is a general LHV theory and satsifies

$$E(\theta, \phi) = \int_\lambda \rho(\lambda) d\lambda E(\theta|\lambda) E(\phi|\lambda)$$

where $E(\theta|\lambda)$ is the expected value of the spin at Alice's location and similarly for $E(\phi|\lambda)$. The locality assumption is that the $E(\theta|\lambda)$ does not depend on ϕ and $E(\phi|\lambda)$ does not depend on θ, and there is the *factorisation* in the integrand. Check literature to see the proof in this case.

STEP 3

Exercise 4: check your QM prediction for the case: $\theta = 0$, $\phi = \pi/4$, $\theta' = \pi/2$, $\phi' = 3\pi/4$

The maximum violation of the Bell inequality possible algebraically is 4, but by QM is ("Tsirelson bound") $B = 2\sqrt{2}$.

STEP 4

Conclusion: QM and LHV give *different* predictions for a simple entangled state. Experiment needs to check which is right.

1.5 Clauser, Aspect, Zeilinger experiments

Most experiments on entanglement and nonlocality so far have dealt with photons. The first most famous realisations of Bell theorem tests are from Clauser, Aspect, Zeilinger and their many colleagues. We examine in the lecture experiments similar to that performed by Aspect and Zeilinger with polarised correlated photons. The source was initially two-photon atomic cascade, but later the twin output beasm of the optical parametric oscillator was used.

The correlated polarised photon source is now used routinely as a source of "qubit" entanglement- qubit meaning two values eg spin up or down, in this case the two values are photon either polarised along or orthogonal to an axis.

Introduce some terminology for this.

Quantisation of the radiation field / harmonic oscillator

A mode of the field is quantised as a harmonic oscillator:

$$H = \hbar\omega(a^\dagger a + 1/2)$$

where $a^\dagger$,a are creation and destruction operators $[a^\dagger, a] = 1$. The $n = a^\dagger a$ is the (photon) number operator (we sometimes drop the "hat" if meaning of operator is clear) , and we can define eigenstates of this number operator $\hat{n}|n\rangle = n|n\rangle$. The vacuum state is $|0\rangle$ and raising lowering operator rules apply: $a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle$ $a|n\rangle = \sqrt{n}|n-1\rangle$. So, we can use symbols, $|0\rangle$, $|1\rangle$ to refer to spin Up or Down, OR a single or zero excitation of a mode OR whether a photon occupies polarisation mode $+$ or $-$. The most common qubit is a photon in $+$ polarised mode (bit value $+1$) versus photon in $-$ polarised mode (bit value 0). Remember, the orthogonal polarisations correspond to different field modes.

Common simple approach: to describe light through a **beam splitter** (50/50 mirror) OR **polariser** : creation of rotated modes

$$\begin{aligned} a_{out,+} &= \cos\theta a_+ + \sin\theta a_- \\ a_{out,-} &= -\sin\theta a_+ + \cos\theta a_- \end{aligned}$$

Check that the photon number conserved-

Exercise 3: Evaluate the correlation for the output state

$$|\psi\rangle = \frac{1}{\sqrt{2}}\{|1\rangle_{a+}|0\rangle_{a-}|1\rangle_{b+}|0\rangle_{b-} + |0\rangle_{a-}|1\rangle_{a+}|0\rangle_{b+}|1\rangle_{b_-}\}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}}\{a_+^\dagger b_+^\dagger + a_-^\dagger b_-^\dagger\}|0\rangle|0\rangle|0\rangle|0\rangle \\
&= \frac{1}{\sqrt{2}}\{\{\cos\theta a_{out,+}^\dagger - \sin\theta a_{out,-}^\dagger\} \\
&\quad \times\{\cos\phi b_{out,+}^\dagger - \sin\phi b_{out,-}^\dagger\} \\
&\quad +\{\sin\theta a_{out,+}^\dagger + \cos\theta a_{out,-}^\dagger\} \\
&\quad \times\{\sin\phi b_{out,+}^\dagger + \cos\phi b_{out,-}^\dagger\}\}|0\rangle|0\rangle \\
&= \frac{1}{\sqrt{2}}\{\cos(\theta-\phi)\{|1\rangle_{out,a+}|1\rangle_{out,b+} + |1\rangle_{out,a-}|1\rangle_{out,b-}\} \\
&\quad + \sin(\theta-\phi)\{|1\rangle_{out,a+}|1\rangle_{out,b-} - |1\rangle_{out,a-}|1\rangle_{out,b+}\}
\end{aligned}
$$

where we have abbreviated $|1\rangle_{out,a+}|1\rangle_{out,b+} \equiv |1\rangle_{out,a+}|0\rangle_{out,a-}|1\rangle_{out,b+}|0\rangle_{out,b-}$ etc.

Evaluation of the "spin product": probability of spin product being $+1$ is $\cos^2(\theta-\phi)$; probability of product being -1 is $\sin^2(\theta-\phi)$. Hence

$$E(\theta,\phi) = \cos 2(\theta-\phi)$$

Exercise 4: Show how this result gives a violation of the CHSH Bell inequality (hint, take the values for ths spin case in lecture slides and divide by 2.

This is the prediction for the photon polarisation experiments: Bell inequality is violated.

Tests of nonlocality have now expanded into other regimes, eg there are demonstrations of EPR paradox and entanglement for "CV" systems, where observables have a continuous eigenvalue spectrum. These are exlained in the slides, and discussed further below. So far however, for CV systems, there has been no demonstration of a violation of a Bell inequality.

1.6 GHZ " all or nothing" multiparty nonlocality

The former Bell inequality relies on statistical collection of measurements. Greenberger-Horne-Zeilinger (GHZ) came up with a scenario in which the contradiction between QM and LR can be made for *one* measurement (based on previous correlations).

Consider the three "party" spin state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\uparrow\uparrow\rangle - |\downarrow\downarrow\downarrow\rangle]$$

STEP 1:

Exercise 5

Now consider the product of results for spin (outcome is either ±1) if we measure two y spins and one x spin?:

What is the QM prediction?

$\langle\sigma_x^1\sigma_y^2\sigma_y^3\rangle = \langle\sigma_y^1\sigma_x^2\sigma_y^3\rangle = \langle\sigma_y^1\sigma_y^2\sigma_x^3\rangle = +1$ *always* the product is +1
STEP 2
Secret sharing:

So if Alice and Bob make their measurement of σ_x and σ_y, they can predict that of Charlie's σ_y. vv and etc

So, if EPR's Local Realism (LR) is correct, each spin is described by a hidden variable eg λ_x^1 which assumes the value +1 *or* −1.

STEP 3

The hidden variables are such that $\lambda_x^1\lambda_y^2\lambda_y^3 = +1$

So consider the prediction by LR for the product (what is vaue of $(\lambda_x^k)^2$?) −always, this equals 1. So:

$$\begin{aligned}\langle\sigma_x^1\sigma_x^2\sigma_x^3\rangle &= \langle\lambda_x^1\lambda_x^2\lambda_x^3\rangle \\ &= \langle\lambda_x^1\lambda_y^2\lambda_y^3\ \ \lambda_y^1\lambda_x^2\lambda_y^3\ \ \lambda_y^1\lambda_y^2\lambda_x^3\rangle \\ &= +1\end{aligned}$$

Exercise: Now evaluate the quantum prediction for $\langle\sigma_x^1\sigma_x^2\sigma_x^3\rangle$? You will get −1! This is the opposite to the LR prediction! See Mermin's Physics Today article (on line) for a full explanation.

1.7 Squeezing and squeezing EPR entanglement criteria

Entanglement and nonlocality can be measured with respect to observables that have continuous variable outcomes eg position and momentum. there have been entanglement and nonlocality experiments involving such observables, and most of these have been carried out for optical amplitues. We revise the harmonic oscillator formalism and discuss in lectures the ways to measure such effects.

1.8 Continuous variable (cv) squeezing

Consider harmonic oscillator:

$$\begin{aligned}X &= a + a^\dagger \\ P &= (a^\dagger - a)/i\end{aligned}$$

Then the uncertainty relation follows (use $[a, a^\dagger] = 1$)

$$\Delta X \Delta P \geq 1$$

The *minimum uncertainty states* are the *coherent* states $|\alpha\rangle$, which are the eigenstates $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, and these give $\Delta X = \Delta P = 1$ and the "*squeezed states*" for which $\Delta X = e^{-r}$, $\Delta P = e^r$.

We have "squeezing" when

$$\Delta X < 1$$

Squeezing first observed for light for X (quadrature phase amplitudes) 1980's.

How is this "phase" squeezing measured?

Combine with large coherent field (laser) using a beam splitter (50/50 mirror) to get a measure of this fluctuation eg

$$\begin{aligned} a_{out,+} &= [a_+ + a_-]/\sqrt{2} \\ a_{out,-} &= [-a_+ + a_-]/\sqrt{2} \end{aligned}$$

but if a_+ is very large, it can be classical amplitude $Ee^{-i\theta}$- then the photon number difference between the two arms of the beam splitter is

$a^\dagger_{out,+}a_{out,+} - a^\dagger_{in+}a_{in,+} = E(a^\dagger_- e^{i\theta} + a_- e^{-i\theta})$... this becomes X or P depending on the choice of phase θ.

Need to identify the "quantum limit": defined as that for a coherent state, best to take vacuum $|0\rangle$: so measure noise levels with a_- a vacuum, then compare with noise levels when a_- is the squeezed light source.

Sources: squeezed states are generated using qudatirc Hamiltonians: Take the example in the lecture.

$$H = \kappa E(a^{\dagger 2} + a^2)$$

Solve using Heisenberg's equations of motion: then we get an equation for a.

$$\dot{a} = \frac{2i\kappa E}{\hbar} a^\dagger$$

and the conjugate equation for $a^\dagger$. These two give us an equation of motion for $X = a^\dagger + a$.

$$\dot{X} = -\frac{2E\kappa}{\hbar} P$$

and similarly

$$\dot{P} = -\frac{2E\kappa}{\hbar} X$$

so we define the rotated quadrature $X_{\pi/4} = \frac{1}{\sqrt{2}}(X + P)$ and we see

$$\dot{X}_{\pi/4} = -\frac{2E\kappa}{\hbar} X_{\pi/4}$$

So, we see we have an exponential solution which gives a squeezing in the variance of $X_{\pi/4}$. The variance in the conjugate quadrature $X_{3\pi/4}$ will be increased (check it out). We can change the phase of the squeezed quadrature by introducing a phase for $E\kappa$.

The EPR squeezing correlation between the quadrature amplitudes of Alice and Bob for which $X^A - X^B$ and $P^A + P^B$ are both squeezed (there may be correlaiton between rotated quadratures in the intial case, but one can alter the phase of the pump E and κ to select which quadratures are correlated) can be obtained using the nondegenerate form of the Hamiltonian $H = \kappa E(ab + a^\dagger b^\dagger)$, as can be proved by simply solving the equations of motion in this case. This technique of photon (particle) pair generattion is the current technique for generating CV EPR entanglement. It is also possible to obtain EPR correlations by combining the outputs of two degenerate (single mode) OPO (which are modelled by the single mode quadratic Hamiltonian above) across a beam splitter.

Atomic homodyne has recently been realised in atom optics (see Gross et al , Nature **480**, 219, 2011) where spin changing collisions are used to generate the twin atom beams where an atom pair is generated in m=1 and m=-1 hyperfine states.

1.9 EPR paradox and entanglement experiments

Such CV experiments have detected EPR correlations and entanglement. These are outlined in the lecture material. So far most experiments have been for optical fields, but the above paper comes close to realising a CV entanglement and even an EPR paradox. There are currently a number of proposals to generate EPR state with atoms, and these are very promising.

1.10 Spin squeezing:

Define spins J_x, J_y, J_z:

$$\Delta J_x \Delta J_y \geq |\langle J_z \rangle|/2$$

Spin squeezing when $\Delta J_x < \sqrt{|\langle J_z \rangle|/2}$. Spin squeezing has been measured using optical Schwinger spin (polarisation modes) and more recently for cold atoms. Often, the convention is to take $(\Delta J_z)^2 < |\langle J_x \rangle|/2$, and J_Z is measured as the Schwinger number difference: $J_z = (a_1^\dagger a_1 - a_2^\dagger a_2)/2$, so squeezing shows as a reduced number difference fluctuation.

Two-level atom/ spin formalism

It is useful to revise the Schwinger spin formalism that "creates" a spin system using two boson mode operators.

Let one level (eg of an atom) be denoted $|1\rangle$, the second level $|2\rangle$

define spin operators according to:

$\sigma = |0\rangle\langle 1|$, $\sigma^\dagger = |1\rangle\langle 0|$, $\sigma_z = (|1\rangle\langle 1| - |0\rangle\langle 0|)/2$

If we have a large number N of such atoms: (levels); or two polarisation modes $\pm$ that can be occupied by large number of photons, or two levels that can be occupied by a large number of particles then it extremely useful to

define Schwinger spins (check commutation relations using boson relations: $([a^\dagger, a] = 1)$)...use this to check my relations!)

$$\begin{aligned} J_z &= (a_1^\dagger a_1 - a_2^\dagger a_2)/2 \\ J_x &= (a_1^\dagger a_2 + a_2^\dagger a_1)/2 \\ J_y &= (a_1^\dagger a_2 - a_2^\dagger a_1)/2i \end{aligned}$$

so eg $a_1^\dagger a_1$ is the number of particles occupying level (or "state") 1; and similarly $a_2^\dagger a_2$ is number occupying level 2. Thus J_z gives "number difference" between two levels. If we have a fixed number N of particles, then $a_1^\dagger a_1 +$

$a_2^\dagger a_2$ is conserved as the total number N, which means $j = N/2$ is the "spin" of the system. In some recent eperiments, these levels can be two modes of optical polarisation; two modes of two potential wells of an optical lattice, or two hyperfine atomic states.

Spin squeezing was originated by Ueda and Kitagawa and can be generated where one has a nonlinearity in the Hamiltonian- eg spin squeezing has been realised (Esteve et al, Nature **464**, 1165, 2010) in a two-mode BEC with a two-mode Josephson Hamiltonian $H = \chi' J_Z^2/2 - \kappa J_x/N$ that models two weakly coupled condensates, where κ describes a tunneling rate between them, and χ' the nonlinearity of the BEC (here J_X and J_z represent the Schwinger spin modes above, where a_1 and a_2 correspond to mode operators of the two condensates eg if confined to a potential well. The two mode hamiltonian might also be written $H = \kappa(a^\dagger b + ab^\dagger) + \chi a^{\dagger^2} a^2 + \chi b^{\dagger 2} b^2$. There are many theoretical papers, including that of Ueda and Kitigawa which are referenced, or can be traced back from papers referenced, in th above experimental paper.

2 Formalism of Entanglement

2.1 Pure states

Consider a pure $|\psi\rangle$ for two composite systems A and B - written in terms of a basis set $\sum_{m=-j}^{j} c_m |j,m\rangle$. The state shows entanglement between A and B iff we cannot write the state in the factorised form

$$|\psi\rangle \neq |\psi\rangle_A |\psi\rangle_B$$

where $|\psi\rangle_A$ is a state for Alice's system, similarly $|\psi\rangle_B$ is a state for Bob's system. eg cannot write the singlet $\frac{1}{\sqrt{2}}\{|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B\}$ in this way!

2.2 Mixed states: density operator

More usually a system will be in a mixed state described by a *density operator*:

Recall: A density operator for a pure state $|\psi\rangle$ is the *operator* $|\psi\rangle\langle\psi|$ – here we use bra - ket notation.

So consider spin 1/2 system.

Question: Suppose the system is in $|\psi\rangle = |\uparrow\rangle \equiv |1/2, 1/2\rangle$ – what is ρ? :

Answer: Express ρ in spinor basis $|1\rangle \equiv |\uparrow\rangle$, $|2\rangle \equiv |\downarrow\rangle$, so $\rho_{ij} = \langle i|\rho|j\rangle$

$$\rho \quad = \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Question: What is ρ for superposition? $\quad |\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A - |\downarrow\rangle_A)$

Answer:

$$\rho \quad = \frac{1}{2} \quad \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

Note the off-diagonal elements associated with the superposition!

Question: What is the ρ for the singlet state

$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B)$?

Answer: take suitable basis

$$\rho_{singlet} \quad = \frac{1}{2} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

What is ρ for the 50/50 mixture $|\uparrow\rangle_A|\downarrow\rangle_B$ and $|\downarrow\rangle_A|\uparrow\rangle_B$?

Definition: The density operator for a mixed state ie a state that is in a mixture of pure states $|\psi_R\rangle$ with probability P_R is given as

$$\rho = \sum_R P_R |\psi_R\rangle\langle\psi_R|$$

Question: What is the density matrix for the 50/50 mixture of spin 1/2 $|\uparrow\rangle$ and spin $-1/2$ $|\downarrow\rangle$ states?

$$\rho = \frac{1}{2}[|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|]$$

Answer:

$$\rho = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Exercise: Now evaluate density operator and matrix for the 50/50 mixture of $|\uparrow\rangle_A|\downarrow\rangle_B$ and $|\downarrow\rangle_A|\uparrow\rangle_B$?

Now evaluate matrix ρ for a spin 1/2 system in a 50/ 50 mixture of two superpositions:

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle + |\uparrow\rangle),\ \frac{1}{\sqrt{2}}(|\uparrow\rangle - |\downarrow\rangle).$$

Exercise 7: Prove for all pure states, that $\rho^2 = \rho$. Hence show that for pure state $P = Tr(\rho^2) = 1$. For a mixture $P < 1$.

2.3 Entanglement

We say two systems A and B are separable iff we can express the density operator in the following *factorisable* form:

$$\rho = \sum_R P_R \rho_R^A \rho_R^B$$

where ρ_R^A and ρ_R^B are density operators for system A and B respectively. If this cannot be done, we say the two systems are i*nseparable* or *entangled*.

Exercise: Take the system 50/ 50 mixture of singlet and triplet superpositions:

$$\frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle - |\downarrow\rangle|\uparrow\rangle),\ \frac{1}{\sqrt{2}}(|\uparrow\rangle|\downarrow\rangle + |\downarrow\rangle|\uparrow\rangle).$$

Is it entangled?

Exercise: Consider the "maximally unentangled state" for two spin 1/2 systems (two "qubits"). This is a system in a equal mixture of the composite spin eigenstates.

$$\rho_{noisy} = \frac{1}{4}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \frac{1}{4}\mathbf{I}$$

Consider the system in a mixture of this maximally unentangled state ρ_{noisy} and the Bell singlet state:

$$\rho = p\rho_{singlet} + (1-p)\rho_{noisy}$$

Question: For what p is this state (called a Werner state) entangled?

(This is a fundamental question without an obvious immediate answer- see Peres, Physical Review Letters, method of Positive Partial Transpose PPT)

Exercise: Peres PPT criterion (PRL, 77, 1413 1996)

We write density matrix for Werner mixed state as

$$\rho = \begin{pmatrix} (1-p)/4 & 0 & 0 & 0 \\ 0 & (p+1)/4 & -p/2 & 0 \\ 0 & -p/2 & (p+1)/4 & 0 \\ 0 & 0 & 0 & (1-p)/4 \end{pmatrix}$$

Then write the partial transpose wrt one system only

$$\rho = \begin{pmatrix} (1-p)/4 & 0 & 0 & -p/2 \\ 0 & (p+1)/4 & 0 & 0 \\ 0 & 0 & (p+1)/4 & 0 \\ -p/2 & 0 & 0 & (1-p)/4 \end{pmatrix}$$

Evaluate eignevalues of this new matrix.

If the system is separable (not entangled) then all eigenvalues will be nonnegative. (Peres positive partial transpose condition-PPT).

All eigenvalues are nonnegative except one which is

$$\lambda = -(3p-1)/4$$

so if $p > 1/3$ we definitely have entanglement.

For systems of dimension 2x2 (ie spin 1/2 by spin 1/2), the PPT criterion is necessary and sufficient for entanglement - this was proved later by the Horodecki's (Physics Letter A223, 1, 1996; PRL78, 574 1997; 80,5239,1998). Thus there is no advantage in using the method of Wootter's concurrence which also gives you a necessary and sufficient condition for entanglement but only for 2x2 systems.

Hence we have separability when $p \leq 1/3$ and entanglement when $p > 1/3$.

For higher dimensional systems, it is possible to have states with positive partial transpose that are entangled- this is called BOUND entanglement. BOUND entanglement is not distillable (see slides).

The problem of how to determine whether a ρ is entangled or not is not fully solved. For spin 1/2 systems however $-$ yes, straightforward methods exist eg Peres PPT and Wootter's concurrence method.

2.4 Entanglement measures

How doe we measure entanglement:

$E(\rho) \geq 0$;$E(\rho) = 0$ if state is separable; $E(\rho) = 1$ for Bell states ie those that maximally violate Bell inequalities in spin 1/2 system; is invariant under local operations and classical communication- these cannot be the source of an increased entanglement; entanglement of mixture cannot exceed the sum of the entanglement of components (convex).

Pure state: *Entropy of entanglement E* measures the entanglement of a pure state and is the von Neumann entropy of the *reduced density matrix* defined as

$$\rho^{(A)} = Tr_B|\psi\rangle\langle\psi|$$

ie

$$E(|\psi\rangle\langle\psi|) = -Tr\rho^{(A)} \log_2 \rho^A \equiv -\sum_i \lambda_i log_2 \lambda_i$$

where λ_i are eigenvalues of ρ (finite dimension).

Exercise: Take the Bell singlet $|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B)$. What is the reduced density matrix for Alice's system? What is its E?

$$\rho^A = \frac{1}{2}[|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|]$$

It is a 50/50 mixture. This is a maximally random mixture- only works that way for pure state because the superposition has equal amplitudes - these equal amplitudes correspond to the Bell states which show (and are the only 2 qubit states that show) a maximum violation of Bell-CHSH inequality. Thus the entropy is a measure of entanglement.

What is the E? (1 taking log base 2)

Mixed states $\rho = \sum_R P_R|\psi_R\rangle\langle\psi_R|$ are more difficult. One measure is the *Entanglement of formation*:

$$E_F = \min_{\{p_R,\psi_R\}} \sum_R P_R E(|\psi_R\rangle\langle\psi_R|)$$

For two qubit systems, this can be worked out, and expressed as the "concurrence" - Wootters. For higher dimensions the question of an entanglement measure is a difficult one.

However, we will now look at some entanglement criteria: conditions that are only satisifed if a system is entangled - these are criteria for entanglement that are *sufficient*, but not *necessary* ie they don't pick up all entanglement, but in many cases are easy to calculate and to measure. We will focus on criteria using uncertainty relations and spin observables.

2.5 Applications

It was once considered that entanglement was relevant only in a fundamental sense- however field quantum information has emerged with a different point of view: entanglement is a resource that can be used for applications eg quantum cryptography, quantum teleportation .... see slides for brief summary.

2.6 Local Uncertainty Relations

Squeezing-type entanglement criteria

We can use uncertainty relations to derive criteria **sufficient** to deduce entanglement.

Criterion 1: CV case

Quadrature amplitudes X and P

Assume separability, that the system can be described as a mixture of factorizable states, so that

$$\rho = \sum_R P_R \rho_R^A \rho_{R...}^B$$

where P_R is a probability ($\sum_R P_R = 1$) and ρ_R^A is a quantum density operator for a state at site A, and ρ_R^B one for site B, etc. We follow approach of Duan et al (Physical Review Letters (PRL), 2000) and Hofman and Takeuchi (see below for reference) to derive criteria following from this assumption, that are then criteria *sufficient* (but not necessary) to demonstrate entanglement.

Assuming separability, we can write that the variance of a mixture must not be less than the average of the variances of its components. So if separability holds (no entanglement), we must have

$$\begin{aligned}
\Delta^2(X_A - X_B) + \Delta^2(P_A + P_B) &\geq \sum_R P_R(\Delta_R^2(X_A - X_B) + \Delta_R^2(P_A + P_B)) \\
&= \sum_R P_R[\langle X_A^2\rangle_R + \langle X_B^2\rangle_R - 2\langle X_A\rangle_R\langle X_B\rangle_R] \\
&\quad + \sum_R P_R[\langle P_A^2\rangle_R + \langle P_B^2\rangle_R + 2\langle P_A\rangle_R\langle P_B\rangle_R] \\
&\quad - \sum_R P_R\langle X_A - X_B\rangle_R^2 - \sum_R P_R\langle P_A + P_B\rangle_R^2 \\
&= \sum_R P_R(\Delta_R^2 X_A + \Delta_R^2 P_A + \Delta_R^2 X_B + \Delta_R^2 P_B).
\end{aligned}$$

Here we use that the subscript R denotes the variance or average for the state depicted by R (namely ρ_A^R or ρ_B^R). Note that separability implies the factorisation $\langle X_A X_B\rangle_R = \langle X_A\rangle_R\langle X_B\rangle_R$, and hence the simplification. We use the result that the variance of a mixture cannot be less than the average variance of its components.

Now for a quantum state, the following uncertainty relation follows from $\Delta X \Delta P \geq 1$. So the "local uncertainty relation" is

$$\Delta^2 X + \Delta^2 P \geq 2$$

The inequality then becomes

$$\Delta^2(X_A - X_B) + \Delta^2(P_A + P_B) \geq 4$$

which gives

$$\Delta^2(X_A - X_B) + \Delta^2(P_A + P_B) < 4$$

as a sufficient criterion for entanglement (note the different forms that appear in the literature depending on the choice of normalisation for the definition of X and P).

Note that quantum mechanics allows the Left hand side to be zero, because the commutator of $X_A - X_B$ and $P_A + P_B$ is zero!

Spin criterion 1:

Another uncertainty relation for fixed spin j is (Hofman and Takeuchi, Physical Review A,68, 032103 (2003))

$$(\Delta J_x^k)^2 + (\Delta J_y^k)^2 + (\Delta J_z^k)^2 \geq j$$

and from ths we can derive criteria for entanglement (Hofman and Takeuchi, PRA,68, 032103 (2003)). Consider two systems A and B: Define collective spin observables

$$J_x = J_x^A \pm J_x^B$$

If we have a separable state (no entanglement), then $\rho = \sum_R P_R \rho_R^A \rho_R^B$.

Now, because the variance of a mixture can never be less than the average variance of its components, and then because for a factorised state $\rho_R^A \rho_R^B$,

$$\begin{aligned} \Delta(J_x^A \pm J_x^B) &= \langle (J_x^A \pm J_x^B)^2 \rangle - \langle (J_x^A \pm J_x^B) \rangle^2 \\ &= (\Delta J_x^A)^2 + (\Delta J_x^B)^2 \end{aligned}$$

and after using the Local Uncertainty Relation (LUR) we find that separability implies

$$\begin{aligned} (\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 &\geq \sum_R P_R \{(\Delta J_x)_R^2 + (\Delta J_y)_R^2 + (\Delta J_z)_R^2 \\ &= \sum_R P_R \{(\Delta J_x^A)_R^2 + (\Delta J_y^A)_R^2 + (\Delta J_z^A)_R^2 \\ &\quad + \{(\Delta J_x^B)_R^2 + (\Delta J_y^B)_R^2 + (\Delta J_z^B)_R^2 \\ &\geq 2j \end{aligned}$$

Thus, if

$$(\Delta J_x)^2 + (\Delta J_y)^2 + (\Delta J_z)^2 < 2j$$

then the two systems A and B are entangled.

Exercise: consider the Bell singlet state

$$|\psi\rangle = \frac{1}{\sqrt{2}}[|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B]$$

Will this criterion pick up the entanglement of the Bell state? what is the sum of the variances? Remember the Bohm EPR paradox, for which spins were correlated.

Answer: recall your answer to exercise 1... the EPR paradox. The spins are perfectly correlated in all directions, so the sum of the variances (with appropriate choice of a "sum" or "difference" depending on whether there is the correlation or anticorrelation) is 0.

Exercise: Does this criterion pick up the entanglement of the Werner state mixture? (see article in PRA by Hofman and Takeuchi). Answer, yes the criterion involving 3 spin observables detects entanglement for p>1/3. You should get for the variances (which are all equal for the singet state)in terms of the Pauli spins: $\Delta^2(\sigma_x^A + \sigma_x^B) = 2(1-p)$ which gives the result of entanglement when $p > 1/3$. For the two spin case in lecture slides, the result is not as senstiive, (the criterion only detects entanglement when $p > 1/2$).

Criterion 2:

One can use the product uncertainty relation in a similar way: entanglement is detected if (Giovannetti et al. PRA67, 022320 (2003)).

$$(\Delta J_x)(\Delta J_y) < [|\langle J_z^A\rangle| + |\langle J_z^B\rangle|]/2$$

Also one can derive the following relation involving the sums of spin varainces.

Assume separability (no entanglement): then, using same procedure as above,

$$\begin{aligned}\Delta^2(J_x^A \mp J_x^B) + \Delta^2(J_y^A \pm J_y^B) &\geq \sum_R P_R \Delta_R^2(J_x^A \mp J_x^B) + \sum_R P_R \Delta_R^2(J_y^A \pm J_y^B) \\ &= \sum_R P_R(\Delta_R^2 J_x^A + \Delta_R^2 J_y^A + \Delta_R^2 J_x^B + \Delta_R^2 J_y^B).\end{aligned}$$

Now for a quantum state, the following uncertainty relation follows from $\Delta J_x \Delta J_y \geq |\langle J_z\rangle|/2$.

$$\Delta^2 J_x + \Delta^2 J_y \geq |\langle J_z\rangle|$$

(since $(x-y)^2 = x^2 + y^2 - 2xy \geq 0$). If we assume quantum state at both sites, we are able to substitute this quantum bound to derive

$$\begin{aligned}\Delta^2(J_x^A \mp J_x^B) + \Delta^2(J_y^A \pm J_y^B) &\geq \sum_R P_R(\Delta_R^2 J_x^A + \Delta_R^2 J_y^A + \Delta_R^2 J_x^B + \Delta_R^2 J_y^B). \\ &\geq \sum_R P_R[|\langle J_z^A\rangle|_R + |\langle J_z^B\rangle|_R] \\ &\geq |\langle J_z^A\rangle| + |\langle J_z^B\rangle|\end{aligned}$$

Thus a criterion sufficient for entanglement is

$$\Delta^2(J_z^A \mp J_z^B) + \Delta^2(J_y^A \pm J_y^B) < |\langle J_z^A\rangle| + |\langle J_z^B\rangle|$$

This criterion is like that used in "polarisation spin squeezing" experiments.

We define the Schwinger representation, so that $J_x^A = \left(a_+^\dagger a_- + a_-^\dagger a_+\right)/2$, $J_y^A = i\left(a_-^\dagger a_+ - a_+^\dagger a_-\right)/2$, $J_z^A = \left(a_+^\dagger a_+ - a_-^\dagger a_-\right)/2$,$N_A = a_+^\dagger a_+ + a_-^\dagger a_-$. Here a_+ and a_- are mode operators for modes of eg orthogonal polarisation at each site. Equivalent relations will hold for quantum states at sites B etc, and we define similar Schwinger operators for that site. Now we examine the arrangement of Bowen et al (picture in slides), to understand one scenario in which spin squeezing is achieved. Here a_+ , b_+ are large coherent field (local oscillator) replaced by E (real) and much greater than a_- so that the Schwinger operators become $N_A = N_B = E^2$, $J_Z = E^2/2$, $J_X^A = EX_A$, $J_Y = EX_B$. Then we get the "cv realization", of spin squeezing when the beams a_- and b_- are EPR correlated.

2.6.1 Spin squeezing entanglement criterion

(1) It will be useful in proving the spin entanglement criterion to note that for systems of fixed spin, because of the finite dimensionality (eg for spin 1/2 the dimension is 2), there will be a limit to how much squeezing you can get. Consider spin 1/2 sytsem. The outcomes are 1/2 or $-1/2$, so there is a bound on the maximum variance:

$$(\Delta J_y)^2 \leq 1/4$$

which means

$$\Delta J_x \geq |\langle J_z \rangle|$$

so you can't have squeezing here (remember $\langle J_x \rangle \leq 1/2$)! More generally, though

$$(\Delta J_y)^2 \leq j^2$$

so

$$\Delta J_x \geq |\langle J_z \rangle|/2j$$

More squeezing is possible with higher j- (ie more particles).

Full proof:

Consider N spin 1/2 particles (qubits) (Sorenson et al, Nature 409, 63 (2001). Suppose there is no entanglement. Then

$$\rho = \sum_R P_R \rho_R^1 \rho_R^2 .... \rho_R^N$$

Now consider the variance of the collective spin $J_x = \sum_{k=1}^N J_x^k$. for a separable state, this variance is constrained to be above a certain value.

Note for each spin 1/2 subsystem, there is a minimum for $(\Delta J_x^k)^2$ because there is a maximum on $(\Delta J_y^k)^2$ becasue the system has a finite dimension ie resuts $-1/2$ or $1/2$, so

$$(\Delta J_y^k)^2 \leq 1/4$$

Now, using the Heisenberg Uncertainty principle, $\Delta J_x^k \Delta J_y^k \geq |\langle J_z^k \rangle|/2$, so we can deduce always

$$(\Delta J_x^k)^2 \geq |\langle J_z^k \rangle|^2$$

for a spin 1/2 system (Sorenson and Molmer, PRL86, 4431 (2001)). Hence, once can show for a separable state (Sorenson et al, Nature 409, 63 (2001))

$$(\Delta J_x)^2 \geq |\langle J_z \rangle|^2/N$$

which says that if you measure an amount of spin squeezing below a certain level,

$$(\Delta J_x)^2 < |\langle J_z \rangle|^2/N$$

then there must be entanglement in the system.

Proof: Using convexity and separability as before

$$\begin{aligned}
(\Delta J_x)^2 &\geq \sum_R P_R (\Delta J_x)_R^2 \\
&= \sum_R P_R \sum_{k=1}^N (\Delta J_x^k)_R^2 \\
&\geq \sum_R P_R \sum_{k=1}^N \langle J_z^k \rangle_R^2 \\
&= \sum_{k=1}^N \sum_R P_R \langle J_z^k \rangle_R^2 [\sum_R P_R]
\end{aligned}$$

where we have used the Cauchy Scwarz inequality, and $\sum_R P_R = 1$. Now one can use Cauchy Schwarz inequality again, $\langle u^2 \rangle \langle v^2 \rangle \geq |\langle uv \rangle|^2$, $u = \langle J_z^k \rangle_R$, $v = 1$, and again below to get

$$\begin{aligned}
(\Delta J_x)^2 &\geq \sum_{k=1}^N |\sum_R P_R \langle J_z^k \rangle_R|^2 \\
&= \sum_{k=1}^N |\langle J_z^k \rangle|^2 = N \sum_{k=1}^N (1/N) |\langle J_z^k \rangle|^2 [\sum_{k=1}^N 1/N] \\
&\geq N |\sum_{k=1}^N (1/N) \langle J_z^k \rangle|^2 \\
&= |\langle J_z \rangle|^2/N
\end{aligned}$$

which says that if you measure an amount of spin squeezing below a certain level,

$$(\Delta J_x)^2 < |\langle J_z \rangle|^2/N$$

then there must be some entanglement in the system (Sorenson et al, Nature 409, 63 (2001)).

This criterion was used in the experiment Esteve et al, Nature, **455** 1216, 2008 and in another experiment, Gross et al Nature **464** 1165, 2010 the depth of squeezing enabled a deduction of how many atoms were entangled, as summarised in the lecture slides.

2.7 Multipartite entanglement

Bipartite entanglement
So, a generalised Bell state is one like

$$|\psi\rangle = \cos\theta|00\rangle + \sin\theta|11\rangle$$

or

$$|\psi\rangle = \cos\theta|01\rangle + \sin\theta|10\rangle$$

How does entanglement vary with θ? Want a simple measure of entanglement. This is given above, with the entropy of entanglement.

There are four maximally entangled Bell states :

$$|\psi\rangle = \frac{1}{\sqrt{2}}\{|00\rangle \pm |11\rangle\}$$

$$|\psi\rangle = \frac{1}{\sqrt{2}}\{|01\rangle \pm |10\rangle\}$$

Tripartite entanglement
Have two types of pure tripartite entangeld states:
GHZ state (tend to Schrodinger cats, as number of parties become large)

$$|\psi\rangle = \frac{1}{\sqrt{2}}\{|000\rangle + |111\rangle\}$$

or W-state

$$|\psi\rangle = \frac{1}{\sqrt{3}}\{|100\rangle + |010\rangle + |001\rangle\}$$

How to find criteria to detect tripartite entanglement is explained in the slides. Svetlichny first considered the issue of tripartite nonlocality and deried Bell inequalities to test for three body nonlocality (paper reference give in slides). There has been an experimental test of this inequality using photons. For the CV case, criteria for tripartite entanglement have been given by van Loock and Furusawa, Phys Rev A 67, 052315 (2003), and there have also been experimental realisations.