Dimensional reduction in single-component BECs

Michael Egorov

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SWINBURNE
CENTRE FOR ATOM OPTICS AND ULTRAFAST SPECTROSCOPY
AUSTRALIAN CENTRE FOR QUANTUM-ATOM OPTICS

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Order parameter

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- In mean-field approximation all particles in BEC share the same wavefunction $\Psi(\mathbf{r})$ called order parameter;
- $\Psi(\mathbf{r}) = \sqrt{n(\mathbf{r})}\, e^{i\varphi(\mathbf{r})}$;
- $\int \Psi^*(\mathbf{r})\Psi(\mathbf{r})\, d^3\mathbf{r} = N$ (sometimes normalized to 1).

You can use conventional operators...

Momentum operator

$$\hat{p} = -i\hbar\nabla$$

Kinetic energy operator

$$\hat{E}_k = \frac{\hat{p}^2}{2m} = -\frac{\hbar^2\nabla^2}{2m}$$

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Example: kinetic energy of a BEC

$$E_k = \int \Psi^* \hat{E}_k \Psi \, d^3\mathbf{r} = -\int \Psi^* \frac{\hbar^2 \, \nabla^2\Psi}{2m} \, d^3\mathbf{r}$$

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C. Pethick and H. Smith.
Bose-Einstein condensation in dilute gases
Cambridge University Press (2008)

Energy functional

Energy functional of a trapped BEC

$$E = \int \Psi^*(\mathbf{r}) \left(-\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r}) + \frac{1}{2} U \left| \Psi \right|^2 \right) \Psi(\mathbf{r}) \, d^3\mathbf{r}$$

where

- $V(\mathbf{r})$ – trapping potential;
- $U = \frac{4\pi\hbar^2 a}{m}$ – mean-field repulsion, a – s-wave scattering length;
- $\hat{p}^2/2m = (-i\hbar\nabla)^2/2m = -\hbar^2\nabla^2/2m$ – kinetic energy.

Least action principle

Action functional [Pitaevskii, Stringari]

$$S = i\hbar \int \Psi^* \frac{\partial}{\partial t} \Psi \, d^3\mathbf{r} \, dt - \int E \, dt$$

From now, $\Psi(\mathbf{r}) \equiv \Psi$ for simplicity

Least action principle

$$\delta S = 0, \text{ or } \delta S / \delta \Psi^* = 0$$

L.P. Pitaevskii and S. Stringari.
Bose-Einstein condensation
Clarendon press, Oxford (2003)

Solution

$$S = \int \mathcal{L} \, d^3\mathbf{r} \, dt,$$

where Lagrangian density $\mathcal{L}$:

$$\mathcal{L} = i\hbar\Psi^* \frac{\partial}{\partial t}\Psi + \Psi^* \frac{\hbar^2 \nabla^2}{2m}\Psi - \Psi^* V(\mathbf{r})\Psi - \frac{1}{2} U \left|\Psi\right|^2 \Psi^*\Psi$$

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Solution of $\delta S = 0$

$$\frac{\partial \mathcal{L}}{\partial \Psi^*} = 0$$

Single-component BEC

$$\mathcal{L} = i\hbar\Psi^* \frac{\partial}{\partial t}\Psi + \Psi^* \frac{\hbar^2 \nabla^2}{2m}\Psi - \Psi^* V(\mathbf{r})\Psi - \frac{1}{2}U\left|\Psi\right|^2 \Psi^*\Psi$$

- $\frac{\partial}{\partial \Psi^*}\left(-i\hbar\Psi^* \frac{\partial}{\partial t}\Psi\right) = -i\hbar\frac{\partial \Psi}{\partial t};$

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- $\frac{\partial}{\partial\Psi^*}\left(\frac{1}{2}U\left|\Psi\right|^2 \Psi^*\Psi\right) = \frac{\partial}{\partial\Psi^*}\left(\frac{1}{2}U(\Psi^*)^2\Psi^2\right) = U\left|\Psi\right|^2 \Psi.$

Single-component BEC

$$\mathcal{L} = i\hbar\Psi^* \frac{\partial}{\partial t}\Psi + \Psi^* \frac{\hbar^2 \nabla^2}{2m}\Psi - \Psi^* V(\mathbf{r})\Psi - \frac{1}{2}U|\Psi|^2 \Psi^*\Psi$$

- $\frac{\partial}{\partial \Psi^*}\left(-i\hbar\Psi^* \frac{\partial}{\partial t}\Psi\right) = -i\hbar\frac{\partial \Psi}{\partial t}$;
- $\frac{\partial}{\partial \Psi^*}\left(\Psi^* \frac{\hbar^2 \nabla^2}{2m}\Psi\right) = \frac{\hbar^2 \nabla^2 \Psi}{2m}$;
- $\frac{\partial}{\partial \Psi^*}\left(\Psi^* V(\mathbf{r})\Psi\right) = V(\mathbf{r})\Psi$;
- $\frac{\partial}{\partial \Psi^*}\left(\frac{1}{2}U|\Psi|^2 \Psi^*\Psi\right) = \frac{\partial}{\partial \Psi^*}\left(\frac{1}{2}U(\Psi^*)^2\Psi^2\right) = U|\Psi|^2\Psi.$

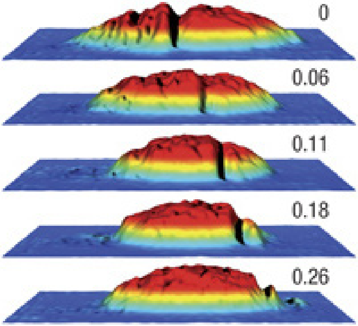
And $\partial\mathcal{L}/\partial\Psi^* = 0$ turns into...

Gross-Pitaevskii equation!

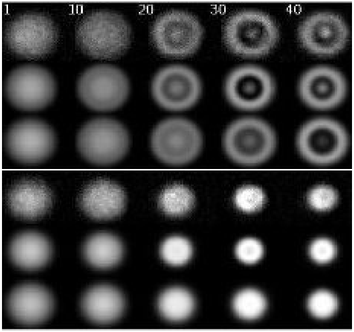
$$i\hbar\frac{\partial \Psi}{\partial t} = \left(-\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r}) + U|\Psi|^2\right)\Psi$$

Often in cigar-shaped BEC dynamics are one-dimensional (or two-dimensional in pancake-shaped BEC)...

(a) Dark solitons in a trapped BEC (Nature Phys. 4, 496–501 (2008))



(b) Ring excitations in a binary BEC (PRL 99, 190402 (2007))



However, BEC is still three-dimensional, and it is hard to solve Gross-Pitaevskii equation analytically!

Factorisation

Need to transform equations $\Psi(x, y, z) \to f(z)$.

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Need to transform equations $\Psi(x, y, z) \to f(z)$.
One of the methods [Salasnich et al.]:

$$\Psi(x, y, z, t) = \phi(x, y, \sigma(z))\, f(z, t),$$

where ϕ is normalized to 1, f can be normalized to N, and

$$\phi(x, y, \sigma(z, t)) = \frac{1}{\pi^{1/2}\sigma(z, t)} e^{-\frac{x^2+y^2}{2\sigma(z,t)^2}}.$$

This already assumes radial gradient of phase $\partial\varphi/\partial r = 0$ and, hence, no radial flow of density.

L. Salasnich, A. Parola, and L. Reatto.
Effective wave equations for the dynamics of cigar-shaped and disk-shaped Bose condensates
Phys. Rev. A, vol. 65, no. 4, 043614 (2002)

Comparison with actual wavefunction

- Exact in a case of non-interacting BEC in harmonic trap;

Comparison with actual wavefunction

- Exact in a case of non-interacting BEC in harmonic trap;
- Differs from Thomas-Fermi BEC profile. However, still works!

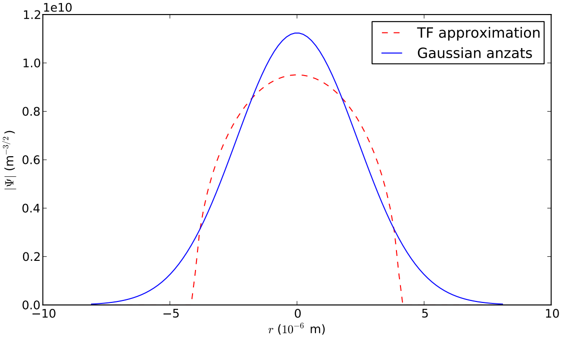
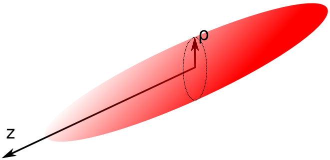


Figure: (Calculated for 10^5 ^{87}Rb atoms in a harmonic trap $100 \times 100 \times 10$ Hz)

Action in cylindrical coordinates

$$S = \iiint \left(i\hbar\Psi^* \frac{\partial}{\partial t}\Psi + \Psi^* \frac{\hbar^2 \nabla^2}{2m}\Psi - \Psi^* V\Psi - \frac{1}{2}U\left|\Psi\right|^2 \Psi^*\Psi \right) 2\pi\rho\, d\rho\, dz\, dt,$$



- $\rho^2 = x^2 + y^2$
- $V = m(\omega_\rho^2\rho^2 + \omega_z^2 z^2)/2$
- $\Psi = \phi(\rho, \sigma(z,t))\, f(z,t)$

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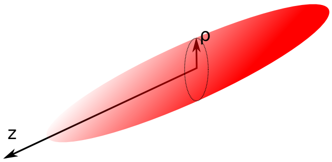

- $\rho^2 = x^2 + y^2$
- $V = m(\omega_\rho^2 \rho^2 + \omega_z^2 z^2)/2$
- $\Psi = \phi(\rho, \sigma(z,t))\, f(z,t)$
- Assumming that ϕ slowly varies along the condensate, i.e. $\nabla^2\phi \approx \nabla_\rho^2 \phi$, where $\nabla_\rho^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 = \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial \rho^2}$

Now, we integrate along radial direction ρ.

Integrating action terms

$$\mathcal{L}_{1\mathrm{D}} = \int_0^\infty \left(i\hbar \Psi^* \frac{\partial}{\partial t}\Psi + \Psi^* \frac{\hbar^2 \nabla^2}{2m}\Psi - \Psi^* V \Psi - \frac{1}{2} U \left|\Psi\right|^2 \Psi^* \Psi \right) 2\pi\rho\, d\rho,$$

$$f \equiv f(z), \quad \phi = \frac{1}{\pi^{1/2}\sigma} e^{-\frac{\rho^2}{2\sigma^2}}, \quad \Psi = \phi f, \quad \int_0^\infty \phi^2\, 2\pi\rho\, d\rho = 1$$

$$\int_0^\infty \left(\phi f^* \frac{\partial(\phi f)}{\partial t} \right) 2\pi\rho\, d\rho =$$

$$f^* f \int_0^\infty \left(\phi \frac{\partial \phi}{\partial t} \right) 2\pi\rho\, d\rho + f^* \frac{\partial f}{\partial t}$$

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$$\int_0^\infty \Psi^* \nabla^2 \Psi\, 2\pi\rho\, d\rho = \int_0^\infty f^* \phi \left(\nabla_\rho^2 + \frac{\partial^2}{\partial z^2} \right) (f\phi)\, 2\pi\rho\, d\rho =$$

$$f^* f \int_0^\infty \phi \nabla_\rho^2 \phi\, 2\pi\rho\, d\rho + f^* \frac{\partial^2 f}{\partial z^2} = -\frac{f^* f}{\sigma^2} + f^* \frac{\partial^2 f}{\partial z^2}$$

Laplace operator is conveniently applied to the factorised wavefunction

Integrating action terms

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$$\int_0^\infty f^* \phi \left(\frac{m\omega_\rho^2 \rho^2}{2} + \frac{m\omega_z^2 z^2}{2} \right) f \phi \, 2\pi\rho \, d\rho =$$

$$f^* f \frac{m\omega_z^2 z^2}{2} + f^* f \frac{m\omega_\rho^2 \sigma^2}{2}$$

Integrating action terms

$$\mathcal{L}_{\mathrm{1D}} = \int_0^\infty \left(i\hbar\Psi^* \frac{\partial}{\partial t}\Psi + \Psi^* \frac{\hbar^2 \nabla^2}{2m}\Psi - \Psi^* V \Psi - \frac{1}{2} U \left|\Psi\right|^2 \Psi^* \Psi \right) 2\pi\rho\, d\rho,$$

$$f \equiv f(z), \quad \phi = \frac{1}{\pi^{1/2}\sigma} e^{-\frac{\rho^2}{2\sigma^2}}, \quad \Psi = \phi f, \quad \int_0^\infty \phi^2\, 2\pi\rho\, d\rho = 1$$

$$\int_0^\infty (\Psi^*)^2\, \Psi^2\, 2\pi\rho\, d\rho = (f^*)^2\, f^2 \int_0^\infty \phi^4\, 2\pi\rho\, d\rho =$$

$$\frac{1}{2\pi\sigma^2} (f^*)^2\, f^2$$

Resulting action functional

$$S = \iint \mathcal{L}_{\mathrm{1D}}\, dz\, dt$$

$$\mathcal{L}_{\mathrm{1D}} = f^* \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} - \frac{m\omega_z^2 z^2}{2} - \frac{U f^* f}{4\pi\sigma^2} - \frac{\hbar^2}{2m\sigma^2} - \frac{m\omega_\rho^2 \sigma^2}{2} \right] f$$

Resulting action functional

$$S = \iint \mathcal{L}_{1\mathrm{D}}\, dz\, dt$$

$$\mathcal{L}_{1\mathrm{D}} = f^* \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} - \frac{m\omega_z^2 z^2}{2} - \frac{U f^* f}{4\pi\sigma^2} - \frac{\hbar^2}{2m\sigma^2} - \frac{m\omega_\rho^2\sigma^2}{2} + i\hbar \int_0^\infty \left(\phi \frac{\partial\phi}{\partial t}\right) 2\pi\rho\, d\rho \right] f$$

Note: term $i\hbar \int_0^\infty \left(\phi \frac{\partial\phi}{\partial t}\right) 2\pi\rho\, d\rho$ is implicitly assumed to be equal to zero resulting from ϕ being real: will see that at the next slide

Euler-Lagrange equations

Imposing stationary condition on S assumes:

$$\frac{\partial \mathcal{L}_{\rm 1D}}{\partial \sigma} = 0, \qquad \frac{\partial \mathcal{L}_{\rm 1D}}{\partial f^*} = 0$$

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$$\frac{\hbar^2}{2m\sigma^3} - \frac{m\omega_\rho^2\sigma}{2} + \frac{U}{4\pi\sigma^3}|f|^2 = i\hbar\frac{\partial}{\partial\sigma}\int_0^\infty\left(\phi\frac{\partial\phi}{\partial t}\right)2\pi\rho\, d\rho$$

Left side is real number, right side is imaginary number (because ϕ is real, σ is real), therefore it should be 0

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$$i\hbar \frac{\partial f}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_z^2 z^2}{2} + \frac{U}{2\pi\sigma^2}|f|^2 + \left(\frac{\hbar^2}{2m\sigma^2} + \frac{m\omega_\rho^2 \sigma^2}{2} \right) \right] f$$

NPSE

Equation for σ is solved algebraically

$$\sigma^2 = a_\rho^2 \sqrt{1 + 2a\,|f|^2}, \quad \text{where} \quad a_\rho = \sqrt{\frac{\hbar}{m\omega_\rho}}$$

Which leads to. . .

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$$\sigma^2 = a_\rho^2 \sqrt{1 + 2a\,|f|^2}, \quad \text{where} \quad a_\rho = \sqrt{\frac{\hbar}{m\omega_\rho}}$$

Which leads to...

Non-polynomial 1D Schrödinger equation (1D NPSE)

$$i\hbar \frac{\partial f}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_z^2 z^2}{2} + \frac{U}{2\pi a_\rho^2}\frac{|f|^2}{\sqrt{1+2a\,|f|^2}} + \frac{\hbar\omega_\rho}{2}\left(\frac{1}{\sqrt{1+2a\,|f|^2}} + \sqrt{1+2a\,|f|^2} \right) \right] f$$

Limit of weak interactions

$|f|^2 \ll 1/a$, or more practical $N \ll 2R_{\mathrm{TF},z}/a$:

$$\sigma = a_\rho$$

and the BEC is one-dimensional (1D Gross-Pitaevskii equation):

$$i\hbar\frac{\partial f}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_z^2 z^2}{2} + \frac{U}{2\pi a_\rho^2}|f|^2\right] f$$

Limit of strong interactions

$|f|^2 \gg 1/a$ (or $N \gg 2R_{\mathrm{TF},z}/a$):

$$\sigma^2 = \sqrt{2a}\, a_\rho^2\, |f|$$

1D NPSE for 3D cigar-shaped BEC:

$$i\hbar\frac{\partial f}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_z^2 z^2}{2} + \frac{3}{2}\frac{U}{2\pi a_\rho^2\sqrt{2a}}\,|f|\right] f$$

Thomas-Fermi approximation

$$|f|^2 = \frac{2}{9}\frac{1}{(\hbar\omega_\rho)^2 a}\left[\mu' - \frac{m\omega_z^2 z^2}{2}\right]^2$$

where chemical potential μ' is obtained using normalisation condition:

$$\mu' = \left(\frac{135 N a \hbar^2 \omega_r^2 \omega_z \sqrt{m}}{2^{\frac{11}{2}}}\right)^{\frac{2}{5}}$$

Radial Gaussian profile is an approximation, so the usual 3D TF approximation gives a different result, where μ is smaller by 4.6%!

$$|f_{3D}|^2 = \frac{1}{4}\frac{1}{(\hbar\omega_\rho)^2 a}\left[\mu' - \frac{m\omega_z^2 z^2}{2}\right]^2, \quad \mu_{3D} = \left(\frac{15 N a \hbar^2 \omega_r^2 \omega_z \sqrt{m}}{2^{\frac{5}{2}}}\right)^{\frac{2}{5}}$$

Testing accuracy of 1D NPSE

EFFECTIVE WAVE EQUATIONS FOR THE DYNAMICS . . . PHYSICAL REVIEW A **65** 043614

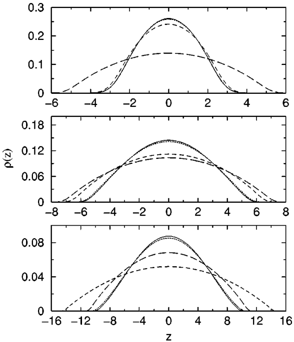

FIG. 1. Normalized density profile $\rho(z)=|f(z)|^2$ along the axial direction z for the cigar-shaped trap. Number of Bosons: $N=10^4$ and trap anisotropy: $\omega_\perp/\omega_z=10$. Four different procedures: 3D GPE (solid line), 1D GPE (dashed line), CGPE (long-dashed line), and 1D NPSE (dotted line). From top to bottom: $a_s/a_z=10^{-4}$, $a_s/a_z=10^{-3}$, and $a_s/a_z=10^{-2}$. Length z in units of a_z and density in units of a_z^{-1}.

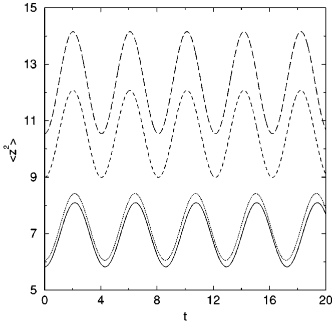

FIG. 2. Squared amplitude $\langle z^2\rangle$ as a function of time t. Number of Bosons: $N=10^4$ and trap anisotropy: $\omega_\perp/\omega_z=10$. Four different procedures: 3D GPE (solid line), 1D GPE (dashed line), CGPE (long-dashed line), and 1D NPSE (dotted line). Scattering length: $a_s/a_z=10^{-3}$. Length z in units of a_z, density in units of a_z^{-1}, and time t in units of $1/\omega_z$.

2D reduction: Factorisation and action functional

$$\Psi(x,y,z) = \phi(x,y)\, f(z,\eta(x,y)), \quad f = \frac{1}{\pi^{1/4}\eta^{1/2}} e^{-\frac{z^2}{2\eta^2}}$$

Action:

$$S = \iiint \mathcal{L}_{\mathrm{2D}}\, dx\, dy\, dt$$

Lagrangian density:

$$\mathcal{L}_{\mathrm{2D}} = \phi^* \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla_\perp^2 - \frac{m\omega_\rho^2\left(x^2+y^2\right)}{2} - \frac{U}{2\eta\sqrt{2\pi}}|\phi|^2 - \frac{\hbar^2}{4m\eta^2} - \frac{m\omega_z^2\eta^2}{4} \right] \phi$$

where $\nabla_\perp^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$.

2D reduction: result

$$\frac{\partial \mathcal{L}_{\mathrm{2D}}}{\partial \eta} = 0, \quad \frac{\partial \mathcal{L}_{\mathrm{2D}}}{\partial \phi^*} = 0$$

$$i\hbar \frac{\partial \phi}{\partial t} = \left[\frac{\hbar^2}{2m} \nabla_\perp^2 + \frac{m\omega_\rho \left(x^2 + y^2\right)}{2} + \frac{U}{\sqrt{2\pi}\,\eta} |\phi|^2 + \left(\frac{m\omega_z^2 \eta^2}{4} + \frac{\hbar^2}{4m\eta^2} \right) \right] \phi$$

Warning: mistake in [Salasnich *et al.*, 2002]

Useful articles for a single-component case

L. Salasnich, A. Parola, and L. Reatto.
Effective wave equations for the dynamics of cigar-shaped and disk-shaped Bose condensates
Phys. Rev. A, vol. 65, no. 4, 043614 (2002)

A. Kamchatnov and V. Shchesnovich.
Dynamics of Bose-Einstein condensates in cigar-shaped traps
Phys. Rev. A, vol. 70, no. 2, p. 023604 (2004).

L. Young-S., L. Salasnich, and S. Adhikari.
Dimensional reduction of a binary Bose-Einstein condensate in mixed dimensions
Phys. Rev. A, vol. 82, no. 5, p. 053601 (2010).

What's next?

- Two-component 1D equations (cigar-shaped trap)
- Approximate analytic solution
- Applying results to two-component dynamics