Dimensional reduction in two-component BECs

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Collective oscillations

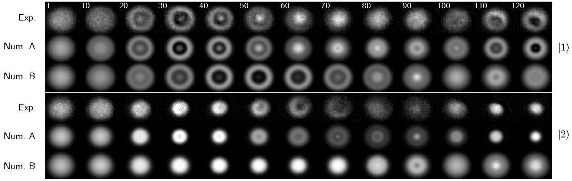


Figure: Ring excitations in a binary BEC. Well predicted by coupled Gross-Pitaevskii equations, but no analytical description and no formula for collective oscillations frequency.

K. M. Mertes, J. W. Merrill, R. Carretero-Gonzlez, D. J. Frantzeskakis, P. G. Kevrekidis, and D. S. Hall
Nonequilibrium dynamics and superfluid ring excitations in binary Bose-Einstein condensates
Physical Review Letters 99, 190402 (2007).

Dephasing and rephasing

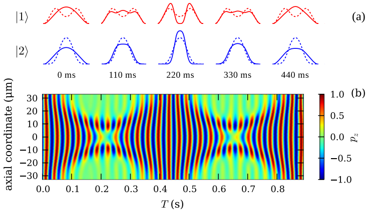


Figure: Periodic revivals of BEC phase coherence are observed. The period coincides with the collective oscillations period. Again, the period of revivals is needed to be found!

M. Egorov, R. P. Anderson, V. Ivannikov, B. Opanchuk, P. Drummond, B. V. Hall, and A. I. Sidorov
Long-lived periodic revivals of coherence in an interacting Bose-Einstein condensate
Phys. Rev. A 84, 021605 (2011)

Two-component action functional

$$S = \int \left(\mathcal{L}_1 + \mathcal{L}_2 - U_{12} \left|\Psi_1\right|^2 \left|\Psi_2\right|^2 \right) \, d^3\mathbf{r} \, dt,$$

where

$$\mathcal{L}_j = i\hbar \Psi_j^* \frac{\partial}{\partial t} \Psi_j + \Psi_j^* \frac{\hbar^2 \nabla^2}{2m} \Psi_j - V \left|\Psi_j\right|^2 - \frac{1}{2} U_{jj} \left|\Psi_j\right|^4 .$$

and $\Psi \equiv \Psi(\mathbf{r})$, $V \equiv V(\mathbf{r})$, $U_{ij} = \frac{4\pi\hbar^2 a_{ij}}{m}$

Deriving coupled GPE

Stationary point of action functional

$$\frac{\delta S}{\delta \Psi_j^*} = 0, \qquad j = 1, 2$$

which is

$$\frac{\partial}{\partial \Psi_j^*} \left(\mathcal{L}_1 + \mathcal{L}_2 - U_{12} \Psi_1^* \Psi_1 \Psi_2^* \Psi_2 \right) = 0$$

turns into Coupled Gross-Pitaevskii equations:

$$i\hbar \frac{\partial \Psi_1}{\partial t} = \left[-\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r}) + U_{11} |\Psi_1|^2 + U_{12} |\Psi_2|^2 \right] \Psi_1,$$
$$i\hbar \frac{\partial \Psi_2}{\partial t} = \left[-\frac{\hbar^2 \nabla^2}{2m} + V(\mathbf{r}) + U_{12} |\Psi_1|^2 + U_{22} |\Psi_2|^2 \right] \Psi_2.$$

Action functional

In analogy with single-component case, in order to transform 3D equations into 1D, we factorise the wavefunctions:

$$\Psi_1 = \phi(x, y, \sigma_1(z))\, f_1(z) \qquad \Psi_2 = \phi(x, y, \sigma_2(z))\, f_2(z)$$

where

$$\phi(x, y, \sigma_j(z, t)) = \frac{1}{\pi^{1/2}\sigma_j(z, t)} e^{-\frac{x^2+y^2}{2\sigma_j(z,t)^2}}.$$

Note that the wavefunctions are allowed to have different widths which affect their overlap and, hence, interaction strength

Action in cylindrical coordinates

Integrating all terms radially $\int \ldots 2\pi\rho\, d\rho$, where $\rho^2 = x^2 + y^2$, we obtain:

$$\mathcal{L}_{\mathrm{1D}} = \mathcal{L}_{1,\mathrm{1D}} + \mathcal{L}_{2,\mathrm{1D}} - \frac{U_{12}}{\pi(\sigma_1^2 + \sigma_2^2)} f_1^* f_1 f_2^* f_2,$$

where

$$\mathcal{L}_{j,\mathrm{1D}} = f_j^* \left[i\hbar \frac{\partial}{\partial t} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} - \frac{m\omega_z^2 z^2}{2} - \frac{U_{jj}\, f_j^* f_j}{4\pi\sigma_j^2} - \frac{\hbar^2}{2m\sigma_j^2} - \frac{m\omega_\rho^2 \sigma_j^2}{2} \right] f_j$$

And now, coupled Euler-Lagnrange equations can be obtained as:

$$\frac{\partial \mathcal{L}_{\mathrm{1D}}}{\partial f_j^*} = 0, \qquad \frac{\partial \mathcal{L}_{\mathrm{1D}}}{\partial \sigma_j} = 0$$

Coupled Scrödinger equations

This results in following system of four 1D equations:

$$i\hbar\frac{\partial}{\partial t}f_1 = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_z^2 z^2}{2} + \left(\frac{\hbar^2}{2m\sigma_1^2} + \frac{m\omega_\rho^2\sigma_1^2}{2}\right) + \frac{U_{11}}{2\pi\sigma_1^2}\left|f_1\right|^2 + \frac{U_{12}}{\pi\left(\sigma_1^2+\sigma_2^2\right)}\left|f_2\right|^2\right] f_1,$$

$$i\hbar\frac{\partial}{\partial t}f_2 = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_z^2 z^2}{2} + \left(\frac{\hbar^2}{2m\sigma_2^2} + \frac{m\omega_\rho^2\sigma_2^2}{2}\right) + \frac{U_{22}}{2\pi\sigma_2^2}\left|f_2\right|^2 + \frac{U_{12}}{\pi\left(\sigma_1^2+\sigma_2^2\right)}\left|f_1\right|^2\right] f_2,$$

$$-\frac{\hbar^2}{2m}\sigma_1^{-3} + \frac{m\omega_\rho^2\sigma_1}{2} - \frac{1}{2}\frac{U_{11}}{2\pi\sigma_1^3}\left|f_1\right|^2 - \frac{U_{12}\sigma_1}{\pi\left(\sigma_1^2+\sigma_2^2\right)^2}\left|f_2\right|^2 = 0,$$

$$-\frac{\hbar^2}{2m}\sigma_2^{-3} + \frac{m\omega_\rho^2\sigma_2}{2} - \frac{1}{2}\frac{U_{22}}{2\pi\sigma_2^3}\left|f_2\right|^2 - \frac{U_{12}\sigma_2}{\pi\left(\sigma_1^2+\sigma_2^2\right)^2}\left|f_1\right|^2 = 0.$$

Weak interactions: 1D GPE

For $|f_j| \ll 1/a_{jj}$ and $|f_j| \ll 1/a_{ij}$:

$$\sigma_1 = \sigma_2 = \sqrt{\frac{\hbar}{m\omega_\rho}} = a_\rho$$

$$i\hbar\frac{\partial f_1}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_z^2 z^2}{2} + \frac{U_{11}}{2\pi a_\rho^2}\left|f_1\right|^2 + \frac{U_{12}}{2\pi a_\rho^2}\left|f_2\right|^2\right] f_1$$

$$i\hbar\frac{\partial f_2}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_z^2 z^2}{2} + \frac{U_{22}}{2\pi a_\rho^2}\left|f_2\right|^2 + \frac{U_{12}}{2\pi a_\rho^2}\left|f_1\right|^2\right] f_2$$

Strong interactions: still difficult to solve analytically!

$$i\hbar\frac{\partial}{\partial t}f_1 = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_z^2 z^2}{2} + \frac{m\omega_\rho^2\sigma_1^2}{2} + \frac{U_{11}}{2\pi\sigma_1^2}\left|f_1\right|^2 + \frac{U_{12}}{\pi\left(\sigma_1^2+\sigma_2^2\right)}\left|f_2\right|^2\right]f_1,$$

$$i\hbar\frac{\partial}{\partial t}f_2 = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_z^2 z^2}{2} + \frac{m\omega_\rho^2\sigma_2^2}{2} + \frac{U_{22}}{2\pi\sigma_2^2}\left|f_2\right|^2 + \frac{U_{12}}{\pi\left(\sigma_1^2+\sigma_2^2\right)}\left|f_1\right|^2\right]f_2,$$

$$\frac{m\omega_\rho^2\sigma_1}{2} - \frac{1}{2}\frac{U_{11}}{2\pi\sigma_1^3}\left|f_1\right|^2 - \frac{U_{12}\sigma_1}{\pi\left(\sigma_1^2+\sigma_2^2\right)^2}\left|f_2\right|^2 = 0,$$

$$\frac{m\omega_\rho^2\sigma_2}{2} - \frac{1}{2}\frac{U_{22}}{2\pi\sigma_2^3}\left|f_2\right|^2 - \frac{U_{12}\sigma_2}{\pi\left(\sigma_1^2+\sigma_2^2\right)^2}\left|f_1\right|^2 = 0.$$

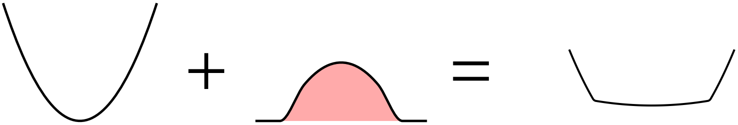
Application to 1D coupled GPE

Approximation $N_2 \ll N_1$, or $|f_2| \ll |f_1|^2$ makes the equations easier.

The idea originally proposed for 1D GPE:

$$i\hbar \frac{\partial f_2}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_z^2 z^2}{2} + \frac{U_{22}}{2\pi a_\rho^2}|f_2|^2 + \frac{U_{12}}{2\pi a_\rho^2}|f_1|^2 \right] f_2$$

Attractive trapping potential is parabolic



$$V = \frac{m\omega^2 z^2}{2} \approx \mu - \frac{U_{11}}{2\pi a_\rho^2}|f_1|^2 \qquad V' = \frac{U_{12}}{2\pi a_\rho^2}|f_1|^2$$

$$\underbrace{\qquad\qquad}_{\mu + \frac{U_{11} - U_{12}}{U_{11}} V}$$

Turn 1D coupled GPE equations into. . .

Application to 1D coupled GPE

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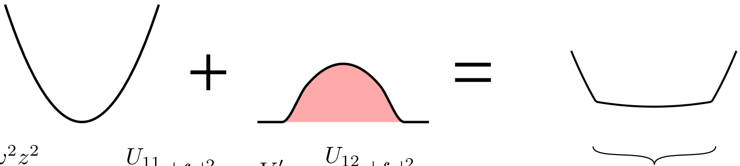
$|f_2|^2$ can be neglected if $N_2 \ll N_1$



$$V = \frac{m\omega^2 z^2}{2} \approx \mu - \frac{U_{11}}{2\pi a_\rho^2}|f_1|^2 \qquad V' = \frac{U_{12}}{2\pi a_\rho^2}|f_1|^2 \qquad \mu + \frac{U_{11} - U_{12}}{U_{11}} V$$

Turn 1D coupled GPE equations into...

Application to 1D coupled GPE

Approximation $N_2 \ll N_1$, or $|f_2| \ll |f_1|^2$ makes the equations easier.

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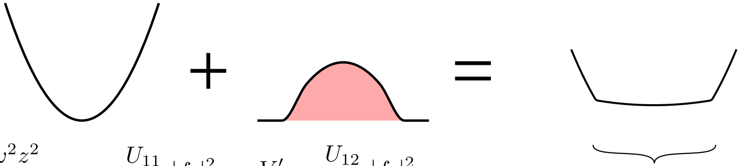
$$i\hbar\frac{\partial f_2}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_z^2 z^2}{2} + \frac{U_{22}}{2\pi a_\rho^2}|f_2|^2 + \frac{U_{12}}{2\pi a_\rho^2}|f_1|^2\right] f_2$$

Repulsive mean-field potential is also parabolic



$$V = \frac{m\omega^2 z^2}{2} \approx \mu - \frac{U_{11}}{2\pi a_\rho^2}|f_1|^2 \qquad V' = \frac{U_{12}}{2\pi a_\rho^2}|f_1|^2 \qquad \mu + \frac{U_{11} - U_{12}}{U_{11}} V$$

Turn 1D coupled GPE equations into. . .

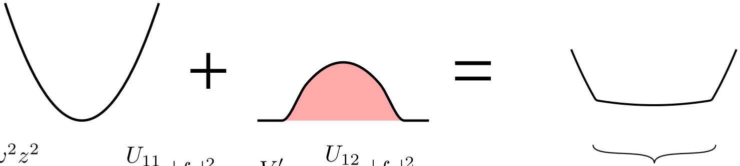
Application to 1D coupled GPE

Approximation $N_2 \ll N_1$, or $|f_2| \ll |f_1|^2$ makes the equations easier.

The idea originally proposed for 1D GPE:

$$i\hbar\frac{\partial f_2}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_z^2 z^2}{2} + \frac{U_{22}}{2\pi a_\rho^2}|f_2|^2 + \frac{U_{12}}{2\pi a_\rho^2}|f_1|^2\right] f_2$$

Therefore, the sum is parabolic!



$$V = \frac{m\omega^2 z^2}{2} \approx \mu - \frac{U_{11}}{2\pi a_\rho^2}|f_1|^2 \qquad V' = \frac{U_{12}}{2\pi a_\rho^2}|f_1|^2 \qquad \mu + \frac{U_{11} - U_{12}}{U_{11}}V$$

Turn 1D coupled GPE equations into. . .

Effective single-component equation

$$i\hbar\frac{\partial f_2}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{a_{11} - a_{12}}{a_{11}}\frac{m\omega_z^2 z^2}{2} + \mu\right] f_2$$

Effectively, a harmonic oscillator!

Note that:

- This works only for $a_{11} > a_{12}$ (for ^{87} Rb $|1\rangle \equiv |F = 1, m_F = -1\rangle$ and $|2\rangle \equiv |F = 2, m_F = 1\rangle$ states, for example). Otherwise, it's a repulsive harmonic potential which fails at the edges of BEC;
- This is in the weak interactions limit!

Z. Dutton and C. Clark
Effective one-component description of two-component Bose-Einstein condensate dynamics
Physical Review A, 71, no. 6, p. 063618 (2005)

Effective single-component eq-s for **strong** interactions

For strong interactions ($|f_1|^2 \gg 1/a_{11}$) **and** $N_2 \ll N_1$, $|f_1|^2$ can be defined using Thomas-Fermi approximation:

$$|f|^2 = \frac{2}{9}\frac{1}{(\hbar\omega_\rho)^2 a_{11}}\left[\mu' - \frac{m\omega_z^2 z^2}{2}\right]^2, \qquad \sigma_1^2 = \frac{\hbar}{2m}\sqrt{2a_{11}}\,|f_1|$$

where μ' is chemical potential. The equations for the component 2 become:

$$\sigma_2^2 = \sigma_1^2\left(2\sqrt{\frac{a_{12}}{a_{11}}} - 1\right)$$

$$i\hbar\frac{\partial f_2}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{4}{3}\left(1 - \sqrt{\frac{a_{12}}{a_{11}}}\right)\frac{m\omega_z^2 z^2}{2} + \frac{\mu'}{3}\left(4\sqrt{\frac{a_{12}}{a_{11}}} - 1\right)\right] f_2$$

Which is also effectively a harmonic oscillator!

Effective single-component equation

So, the approximation $N_2 \ll N_1$ leads to..

Effective harmonic oscillator equation

$$i\hbar\frac{\partial f_2}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_{\text{eff}}^2 z^2}{2} + \mu_{\text{eff}}\right] f_2$$

where

In weak interactions limit $|f_1| \ll 1/a_{11}$

$$\mu_{\text{eff}} = \mu\left(\frac{a_{12}}{a_{11}} - 1\right), \qquad \omega_{\text{eff}} = \omega_z\sqrt{1 - \frac{a_{12}}{a_{11}}}$$

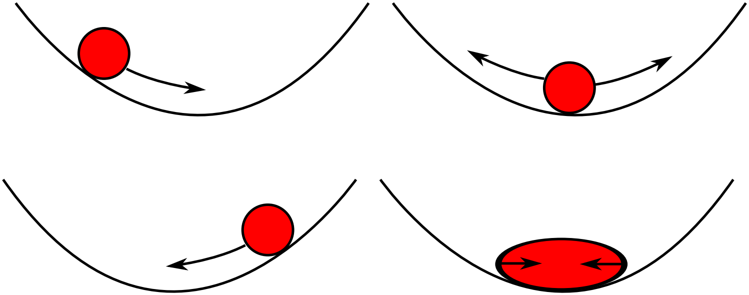
In strong interactions limit $|f_1| \gg 1/a_{11}$

$$\mu_{\text{eff}} = \frac{\mu'}{3}\left(4\sqrt{\frac{a_{12}}{a_{11}}} - 1\right), \qquad \omega_{\text{eff}} = \frac{2}{\sqrt{3}}\sqrt{1 - \sqrt{\frac{a_{12}}{a_{11}}}}\omega_z$$

Collective oscillations

Effective harmonic oscillator equation

$$i\hbar\frac{\partial f_2}{\partial t} = \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + \frac{m\omega_{\rm eff}^2 z^2}{2} + \mu_{\rm eff}\right] f_2$$



(a) Mode $n = 1$ (dipole). Frequency: $f_c = 2\pi/\omega_{eff}$

(b) Mode $n = 2$ (breathing). Frequency: $f_c = 4\pi/\omega_{eff}$

Limits of applicability

Only excited if the relevant harmonic oscillator eigenstates fit into BEC, i.e.

$$R^2_{\mathrm{TF},z} \gg 2\left(n + \frac{1}{2}\right)\frac{\hbar}{m\omega_{\mathrm{eff}}}$$

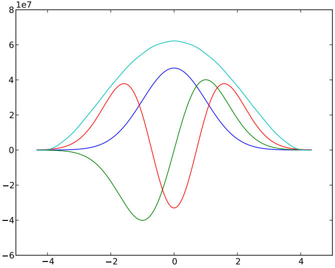
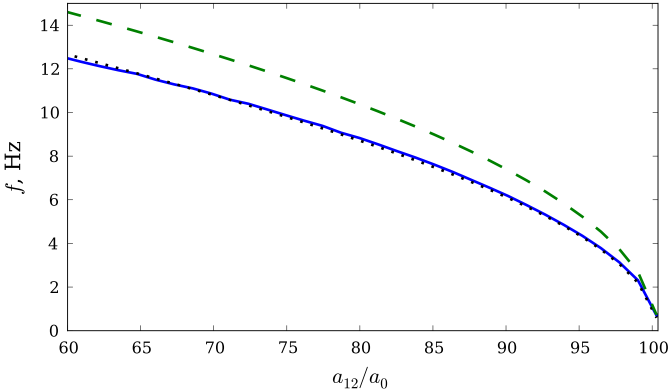


Figure: $n = 0, 1, 2$ states of harmonic oscillator fitting into the BEC density profile

Example: breathing mode frequency

Solid line: 3D coupled GPE;
Dotted line: analytical formula for strong interactions limit;
Dashed line: analytical formula for weak interactions limit.
Experimental parameters: ^{87}Rb, $|1\rangle \equiv |F=1, m_F=-1\rangle$, $|2\rangle \equiv |F=2, m_F=1\rangle$, $100 \times 100 \times 11.507$ Hz trap (Swinburne experiment)



Analytic solution

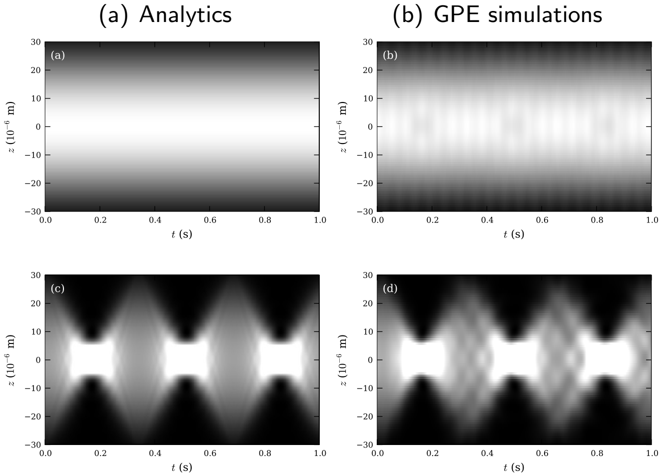
$$i\hbar \frac{\partial f_2}{\partial t} = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \frac{m\omega_{\mathrm{eff}}^2 z^2}{2} + \mu_{\mathrm{eff}} \right] f_2$$

Solution

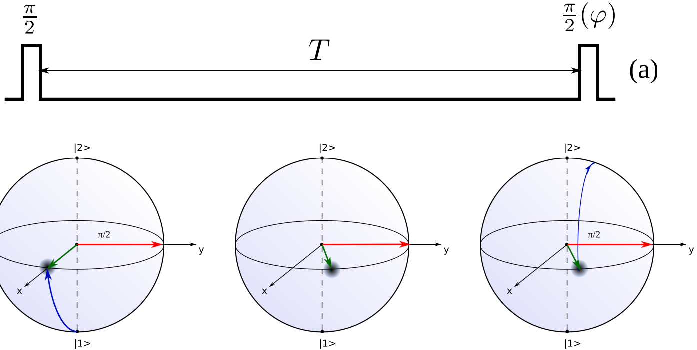
$$f_2(z,t) = e^{-i\mu_{\mathrm{eff}} t/\hbar} \sum_{k=0}^{\infty} \left[e^{-i\omega_{\mathrm{eff}} \left(k+\frac{1}{2}\right) t} \, \psi_{\mathrm{ho}}(k,z) \int \psi_{\mathrm{ho}}(k,\xi)\, f_2(\xi,0)\, d\xi \right],$$

Density evolution

Density of state $|2\rangle$ $|f_2|^2$ (c, d) (and state $|1\rangle$ (a, b)) in a two-component BEC.
Experimental parameters: ^{87}Rb, $|1\rangle \equiv |F = 1, m_F = -1\rangle$, $|2\rangle \equiv |F = 2, m_F = 1\rangle$, $100 \times 100 \times 11.507$ Hz trap (Swinburne experiment)



Ramsey interferometry



$$P_z \equiv \frac{N_1 - N_2}{N_1 + N_2} \propto \cos(2\pi \Delta T + \phi)$$

Δ — detuning of radiation from the transition frequency

ϕ — additional level shifts

Phase evolution

Experimental parameters: ^{87}Rb, $|1\rangle \equiv |F=1, m_F=-1\rangle$, $|2\rangle \equiv |F=2, m_F=1\rangle$, $100 \times 100 \times 11.507$ Hz trap (Swinburne experiment)

$p_z = (n_1 - n_2)/(n_1 + n_2)$

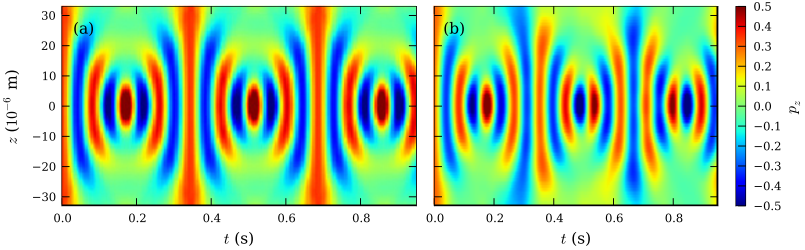


Figure: (a) - anayltics, (b) - GPE simulations

Atom number calibration

Collisional shift ϕ is proportional to chemical potential μ which is proportional to $N^{2/5}$.

$$P_z \equiv \frac{N_1 - N_2}{N_1 + N_2} \propto \cos(2\pi\Delta t + \alpha t N^{2/5})$$

Knowing α, we can find atom number calibration $N_{\mathrm{real}}/N_{\mathrm{measured}}$. Using 1D reduction in strong interactions limit:

$$P_z(N) = A\cos\left[\frac{4}{3\hbar}\left(1 - \sqrt{\frac{a_{12}}{a_{22}}}\right)\left(\frac{135 N a_{11}\hbar^2\bar{\omega}^3\sqrt{m}}{2^{\frac{11}{2}}}\right)^{\frac{2}{5}} t + \varphi_0\right],$$

Solid line: GPE simulations, Dotted line: analytics Dots: experimental data points

Atom number calibration

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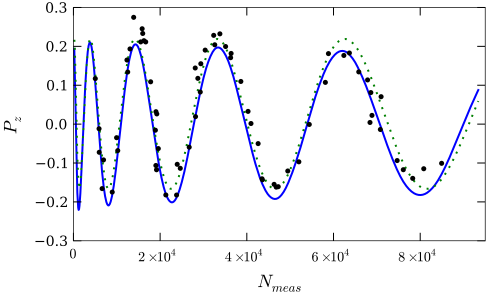
$$P_z(N) = A\cos\left[\frac{4}{3\hbar}\left(1-\sqrt{\frac{a_{12}}{a_{22}}}\right)\left(\frac{135 N a_{11}\hbar^2\bar{\omega}^3\sqrt{m}}{2^{\frac{11}{2}}}\right)^{\frac{2}{5}} t + \varphi_0\right],$$



Solid line: GPE simulations, Dotted line: analytics Dots: experimental data points

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  arXiv:1204.1591, Apr. 2012