Gaussian Phase-Space Representations I

Vssup Lectures 2012

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Outline

Ultracold atoms - the ideal quantum system

ULTRALOW temperatures down to $1nK$

What is different about ultracold atoms?

- Atoms are trapped in a hard vacuum
- Cooling to nanoKelvins or less
- Can have either bosons or fermions
- Atom 'lasers' - atoms behave as quantum objects
- Correlations - mean field theory doesn't always work
- Dynamics - time-evolution is very important

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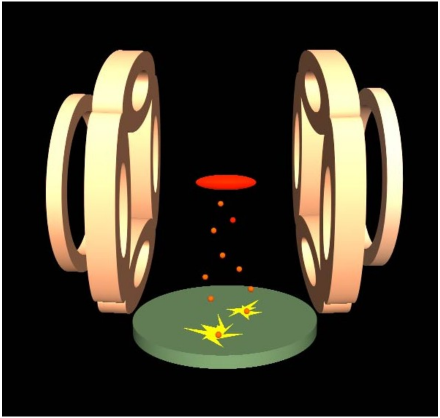
Typical experiment (Orsay, ANU)



How to calculate dynamics?

Classical solution: - use Hamilton's equations

$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$
$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

Quantum mechanics replaces classical quantities by corresponding operators with commutators, so that

$$[\widehat{q}_i, \widehat{p}_j] = i\hbar\delta_{ij}$$
$$[\widehat{q}_i, \widehat{q}_j] = [\widehat{p}_i, \widehat{p}_j] = 0$$

Then, for any operator $\hat{O}$, in the Heisenberg picture:

$$\frac{\partial \hat{O}}{\partial t} = \frac{1}{i\hbar}\left[\widehat{O}, \widehat{H}\right]$$

What about mixtures of states?

Suppose the quantum system is in a mixture of quantum states $|\psi_m\rangle$ with probability p_m . Then the density matrix $\hat{\rho}$ is defined as:

$$\hat{\rho} = \sum_m p_m |\psi_m\rangle \langle\psi_m|$$

In the Schroedinger picture, we let states evolve in time, not operators!

$$\frac{\partial\hat{\rho}}{\partial t} = \frac{1}{i\hbar}\left[\widehat{H}, \widehat{\rho}\right]$$

Then, for any operator $\hat{O}$, the expectation value of the observable is:

$$\left\langle \hat{O} \right\rangle = Tr\left[\hat{\rho}\,\widehat{O}\right]$$

What is the Hamiltonian anyway?

What about the quantum fields with hats?

Recall from the BEC lectures $\widehat{\Psi}_i$ is a quantum field of spin-index i:

$$\left[\widehat{\Psi}_i(\mathbf{x}), \widehat{\Psi}_i^\dagger(\mathbf{x}')\right]_\pm = \delta_{ij}\delta^D(\mathbf{x}-\mathbf{x}')$$

In second quantization the quantum Hamiltonian is

$$\begin{aligned}\widehat{H} &= \sum_i \int d^D\mathbf{x} \left\{ \frac{\hbar^2}{2m} \nabla \widehat{\Psi}_i^\dagger(\mathbf{x}) \cdot \nabla \widehat{\Psi}_i(\mathbf{x}) + V_i(\mathbf{x}) \widehat{\Psi}_i^\dagger(\mathbf{x}) \widehat{\Psi}_i(\mathbf{x}) \right\} \\ &+ \sum_{ij} \frac{U_{ij}}{2} \int d^D\mathbf{x} \widehat{\Psi}_i^\dagger(\mathbf{x}) \widehat{\Psi}_j^\dagger(\mathbf{x}) \widehat{\Psi}_j(\mathbf{x}) \widehat{\Psi}_i(\mathbf{x}) .\end{aligned}$$

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What are the parameters?

This describes a dilute gas at low enough temperatures,

- $\langle\widehat{\Psi}_i^\dagger(\mathbf{x})\widehat{\Psi}_i(\mathbf{x})\rangle$ is the spin i atomic density,
- m is the atomic mass,
- V_i is the atomic trapping potential & Zeeman shift
- U_{ij} is related to the S-wave scattering length in three dimensions by:

$$U_{ij} = \frac{4\pi\hbar^2 a_{ij}}{m}.$$

- Here we implicitly assume a momentum cutoff $k_c << 1/a$

How do we treat quantum fields?

Any field operator $\hat{\Psi}$ can be expanded in orthogonal modes

:

$$\hat{\Psi}(\mathbf{x}) = \sum \hat{a}_m u_m(\mathbf{x})$$

Where: $\int d^3\mathbf{x}\; u_m^*(\mathbf{x})\, u_n(\mathbf{x}) = \delta_{mn}$
Nonvanishing field (anti)-commutators are given by:

$$\left[\hat{\Psi}(\mathbf{x}), \hat{\Psi}^\dagger(\mathbf{x}')\right]_\pm = \delta^3(\mathbf{x}-\mathbf{x}')$$

$(+)$ = anticommutator (FERMION) and
$(-)$ = commutator (BOSON)

(1) Prove as an exercise that: $\left[\hat{a}_m, \hat{a}_n^\dagger\right]_\pm = \delta_{mn}$, $\left[\hat{a}_m, \hat{a}_n\right]_\pm = 0$

What do the mode operators do?

Bosons ↔ harmonic oscillators; fermions ↔ two-level atoms

$$\hat{a}^{\dagger}|N\rangle = \delta_N |N+1\rangle \quad (FERMION)$$
$$\hat{a}^{\dagger}|N\rangle = \sqrt{N+1}|N+1\rangle \quad (BOSON)$$
$$\hat{a}|N\rangle = \sqrt{N}|N-1\rangle$$

Hence the single mode number operator is $\hat{N} = \hat{a}^{\dagger}\hat{a}$:

$$\hat{N}|N\rangle = \hat{a}^{\dagger}\hat{a}|N\rangle = \hat{a}^{\dagger}\sqrt{N}|N-1\rangle = N|N\rangle$$

(2) In the FERMION case, use anticommutators to prove you can only have $N = 0, 1$

What about multi-time correlations?

Suppose we count atoms at multiple times and locations, using delayed coincidences

:

The rate of counting atoms of spins $i_1, \ldots i_m$, at positions and times: $x_1 = (t_1, \mathbf{x}_1), \ldots\ x_m = (t_m, \mathbf{x}_m)$ is:

$$G^{(m)}(x_1, \ldots x_{2m}) = \left\langle \hat{\Psi}^{\dagger}_{i_1}(x_1) \ldots \hat{\Psi}^{\dagger}_{i_m}(x_m) \hat{\Psi}(x_m) \ldots \hat{\Psi}(x_1) \right\rangle$$
$$= Tr\left[\hat{\rho}\hat{\Psi}^{\dagger}(x_1) \ldots \hat{\Psi}^{\dagger}(x_m) \hat{\Psi}(x_m) \ldots \hat{\Psi}(x_1)\right]$$

Note: two-time correlation functions with different arguments like $G^{(1)}(x_1, x_2) = \left\langle \hat{\Psi}^{\dagger}(x_1) \hat{\Psi}(x_2) \right\rangle$ require momentum transfer, eg Bragg scattering, for their measurement.

m-th order coherence

It is useful to define a normalized coherence function as:

$$g^{(m)}(x_1,\ldots x_{2m}) = \frac{G^{(m)}(x_1,\ldots x_{2m})}{\sqrt{\prod_{j=1}^{2m} n(x_j)}}$$

- $n(x_j) = \left\langle \hat{\Psi}^{\dagger}(x_1)\hat{\Psi}(x_1)\right\rangle$ is the counting rate or atom density.
- We say we have complete $m-th$ order coherence if $g^{(m)} = 1$.

What quantum states can we have?

Quantum states are generated from the vacuum state

- Number states:

$$|N_1,\dots N_m\rangle = \frac{\left(a_1^\dagger\right)^{N_1}\dots\left(a_m^\dagger\right)^{N_m}}{\sqrt{N_1!\dots N_m!}}|0\rangle$$

- Properties:

$$\langle \mathbf{M}|\,\mathbf{N}\rangle = \delta_{N_1 M_1}\dots\delta_{N_m M_m}$$

- Fermion case: must have $N_j = 0, 1$ (you just proved this)

All other states can be generated using linear combinations

Example: single mode coherent state

Single mode coherent state has a well-defined phase

- Boson case: Glauber coherent state:

$$|\alpha\rangle = e^{\alpha\hat{a}^{\dagger}-|\alpha|^2/2}|0\rangle = e^{-|\alpha|^2/2}\sum_{N=0}^{\infty}\frac{\alpha^N}{\sqrt{N!}}|N\rangle$$

- Fermion case: Generalized coherent state (see later)

(3) Prove as an exercise:

$$|\langle\alpha|\,\beta\rangle|^2 = e^{-|\alpha-\beta|^2}$$

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$$

$$\langle\alpha|\,\hat{a}^{\dagger}\hat{a}\,|\alpha\rangle = |\alpha|^2$$

Simplest method for state evolution

Suppose the quantum system is described by a few modes:

$$|\psi\rangle = \sum \psi_{\mathbf{N}} |N_1, N_2, \ldots N_m\rangle = \sum \psi_{\mathbf{N}} |\mathbf{N}\rangle$$

Then, let $H_{\mathbf{NM}} = \langle \mathbf{N}| \hat{H} |\mathbf{M}\rangle$ and: $\frac{d}{dt} |\psi\rangle = -\frac{i}{\hbar} \hat{H} |\psi\rangle$
Hence, we have a simple matrix equation:

$$\frac{d}{dt} \psi_{\mathbf{N}} = -\frac{i}{\hbar} \sum_{\mathbf{M}} H_{\mathbf{NM}} \psi_{\mathbf{M}}$$

(4) Prove the last equation using orthogonality

Problem: quantum theory is exponentially complex!

Quantum many-body problems are very large

- consider N particles distributed among M modes
- take $N \simeq M \simeq 500,000$:
- Number of quantum states: $N_s = 2^{2N} = 2^{1,000,000}$
- **More quantum states than atoms in the universe**
- How big is your computer?
- **Can't diagonalize** $2^{1,000,000} \times 2^{1,000,000}$ **Hamiltonian!**

What about losses and damping?

Damping can be treated using a master equation

- The density matrix $\hat{\rho}$ evolves as:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[\hat{H},\hat{\rho}\right] + \sum_j \kappa_j \int d^3\mathbf{r}\mathscr{L}_j\left[\hat{\rho}\right]$$

- Here the Liouville terms describe coupling to the reservoirs:

$$\mathscr{L}_j\left[\hat{\rho}\right] = 2\hat{O}_j\hat{\rho}\hat{O}_j^\dagger - \hat{O}_j^\dagger\hat{O}_j\hat{\rho} - \hat{\rho}\hat{O}_j^\dagger\hat{O}_j$$

- For n-particle collisions: $\hat{O}_i = \left[\widehat{\Psi}_i(\mathbf{r})\right]^n$

Traditional quantum theory methods?

- numerical diagonalisation?
  intractable for $\gtrsim 10$ modes
- operator factorization
  not applicable for strong correlations
- perturbation theory
  diverges at strong couplings
- exact solutions
  not applicable for quantum dynamics

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Quantum theory in classical phase-space

Properties of Wigner/Moyal phase-space

- Maps quantum states into **classical phase-space** $\alpha = p + ix$
- **Wigner first published this representation**
- Moyal showed equivalence to quantum mechanics
- Complexity grows only linearly with number of modes!

Problem: Wigner distribution can have negative values

- **Need to truncate equations to get positive probabilities**

Detailed equivalence

Mapping of characteristic functions

$$W(\boldsymbol{\alpha}) = \frac{1}{\pi^{2M}} \int d^{2M}\mathbf{z} \left\langle e^{i\mathbf{z}\cdot(\hat{\mathbf{a}}-\boldsymbol{\alpha})+i\mathbf{z}^*\cdot\left(\hat{\mathbf{a}}^\dagger-\boldsymbol{\alpha}^*\right)} \right\rangle$$

Operator mean values

- $\left\langle \hat{a}_i^{\dagger m} \hat{a}_j^n \right\rangle_{SYM} = \int d^{2M}\boldsymbol{\alpha}\, \alpha_i^{*m} \alpha_j^n W(\boldsymbol{\alpha}) = \left\langle \alpha_i^{*m} \alpha_j^n \right\rangle_W$
- $\langle \hat{a}_j \rangle = \langle \alpha_j \rangle_W$
- $\left\langle \hat{a}_i^\dagger \hat{a}_j + \hat{a}_i \hat{a}_j^\dagger \right\rangle / 2 = \langle \alpha_i^* \alpha_j \rangle_W$

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Dynamical equivalence

Mapping of dynamical equations

$$\frac{\partial W(\boldsymbol{\alpha})}{\partial t} = \frac{1}{\pi^{2M}} \int d^{2M}\mathbf{z}\, Tr\left[\frac{\partial \hat{\rho}}{\partial t} e^{i\mathbf{z}\cdot(\hat{\mathbf{a}}-\boldsymbol{\alpha})+i\mathbf{z}^*\cdot(\hat{\mathbf{a}}^\dagger-\boldsymbol{\alpha}^*)}\right]$$

Operator mappings

- $\hat{a}_j\hat{\rho} \to \left(\alpha_j + \frac{1}{2}\frac{\partial}{\partial\alpha_j^*}\right) W$
- $\hat{\rho}\hat{a}_j^\dagger \to \left(\alpha_j^* + \frac{1}{2}\frac{\partial}{\partial\alpha_j}\right) W$
- $\hat{a}_j^\dagger\hat{\rho} \to \left(\alpha_j^* - \frac{1}{2}\frac{\partial}{\partial\alpha_j}\right) W$
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Example: Wigner function for a coherent state

Suppose we have a single-mode BEC in a coherent state

$$\hat{\rho} = |\alpha_0\rangle \langle \alpha_0|$$

Hence:

$$W(\alpha) = \frac{1}{\pi^2} \int d^2 z \langle \alpha_0| e^{iz\cdot(\hat{a}-\alpha)+iz\cdot\left(\hat{a}^\dagger-\alpha^*\right)} |\alpha_0\rangle$$

Solution with a little algebra

- $$W(\alpha) = \frac{2}{\pi} e^{-2|\alpha-\alpha_0|^2}$$

(5): show that this solution gives $\langle \alpha^* \alpha \rangle = 1/2$ for a vacuum state

Example: time-evolution of harmonic oscillator

Consider the harmonic oscillator

$$\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}$$

$$\frac{\partial \hat{\rho}}{\partial t} = -i\omega \left[\hat{a}^\dagger \hat{a}\hat{\rho} - \hat{\rho}\hat{a}^\dagger \hat{a}\right]$$

Operator mappings

- $\hat{a}^\dagger \hat{a}\hat{\rho} \to \left(\alpha^* - \frac{1}{2}\frac{\partial}{\partial \alpha}\right)\left(\alpha + \frac{1}{2}\frac{\partial}{\partial \alpha^*}\right) W$
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- $$\frac{\partial W}{\partial t} = i\omega \left(\frac{\partial}{\partial \alpha}\alpha - \frac{\partial}{\partial \alpha^*}\alpha^*\right) W$$

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- $$\frac{\partial W}{\partial t} = i\omega\left(\frac{\partial}{\partial\alpha}\alpha - \frac{\partial}{\partial\alpha^*}\alpha^*\right)W$$

Harmonic oscillator solution

General result for harmonic oscillator

$$\frac{\partial W}{\partial t} = i\omega\left(\frac{\partial}{\partial\alpha}\alpha - \frac{\partial}{\partial\alpha^*}\alpha^*\right)W$$

Solution by method of characteristics

- $$\frac{\partial\alpha}{\partial t} = -i\omega\alpha$$
- $$\alpha(t) = \alpha(0)e^{-i\omega t}$$

(6): Prove this!

Fokker-Planck equations

Result of operator mappings:

$$\frac{\partial W}{\partial t} = \left\{ -\frac{\partial}{\partial \alpha_i} A_i + \frac{1}{2} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j^*} D_{ij} + \frac{1}{6} \frac{\partial^3}{\partial \alpha_i \partial \alpha_j^* \partial \alpha_k^*} T_{ijk} + \ldots \right\} W$$

Scaling to eliminate higher-order terms

$$x = \alpha / \sqrt{N}$$

$$\frac{\partial W}{\partial t} = \left\{ -\frac{1}{\sqrt{N}} \frac{\partial}{\partial x_i} A_i + \frac{1}{2N} \frac{\partial^2}{\partial x_i \partial x_j} D_{ij} + O\left(\frac{1}{N^{3/2}} \right) \right\} W$$

Stochastic equation

Result of operator mappings + truncation - valid if $N/M >> 1$:

$$\frac{\partial W}{\partial t} = \left\{ -\frac{\partial}{\partial \alpha_i} A_i + \frac{1}{2} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j^*} D_{ij} \right\} W$$

Equivalent stochastic equation

$$\frac{\partial \alpha_i}{\partial t} = A_i + \zeta_i(t)$$

where:

$$\left\langle \zeta_i(t) \zeta_j^*(t) \right\rangle = D_{ij} \delta\left(t - t'\right)$$

Example: BEC case

Result of operator mappings + truncation - for the GPE:

$$\frac{d\psi_j}{dt} = iK_j\psi_j - iU_{ij}|\psi_i|^2\psi_j - \gamma_j\psi_j + \sqrt{\gamma_j}\zeta_j(\mathbf{x},t)$$

Here the linear unitary evolution of the wave-function, is described by:

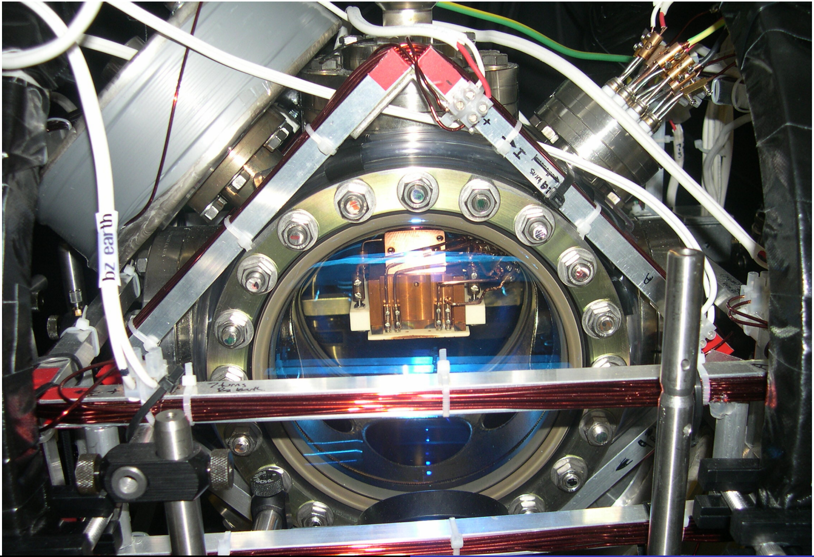
$$K_j = \hbar\nabla^2/2m - V_j(\mathbf{r})$$

while $\zeta_i(\mathbf{x},t)$ is a complex, stochastic delta-correlated Gaussian noise with

$$\left\langle \zeta_i(\mathbf{x},t)\zeta_j^*(\mathbf{x}',t')\right\rangle = \delta_{ij}\delta^3\left(\mathbf{x}-\mathbf{x}'\right)\delta\left(t-t'\right).$$

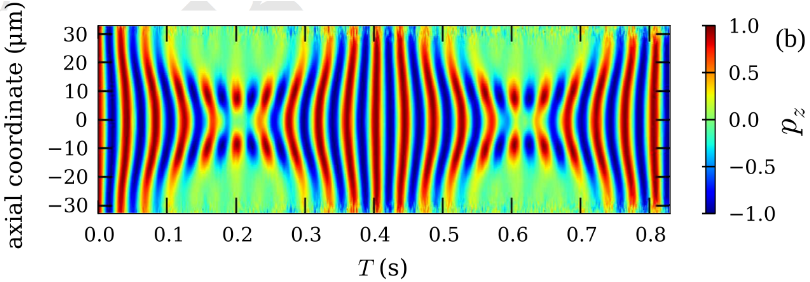
Initial fluctuations: $\langle\Delta\Psi_s(\mathbf{x})\Delta\Psi_u^*(\mathbf{x}')\rangle = \frac{1}{2}\delta_{su}\delta^3(\mathbf{x}-\mathbf{x}')$

Interferometry on an atom chip



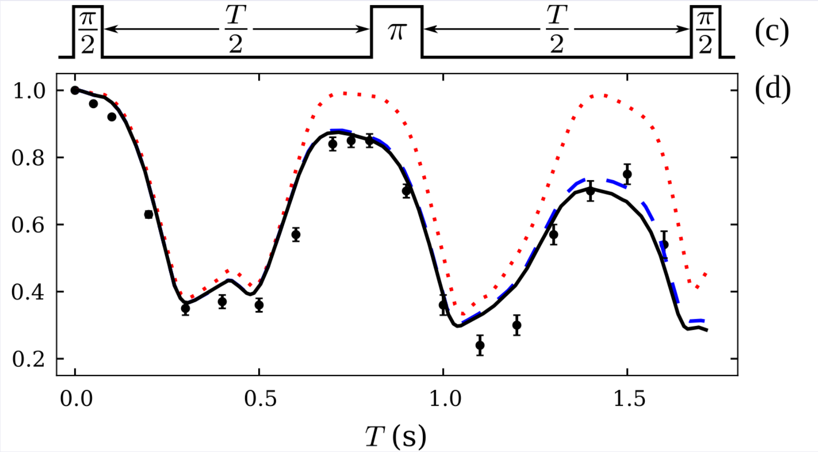
Interferometry

A two-component ^{87}Rb BEC is in a harmonic trap with internal Zeeman states $|1, -1\rangle$ and $|2, 1\rangle$, which can be coupled via an RF field.



Wigner simulations vs BEC fringe visibility

Blue line = Wigner simulation, black line = Wigner + local oscillator noise, red dots = GPE, error bars are measured



SUMMARY

Phase-space representation methods have many applications

Wigner phase-space is relatively simple!

- Maps **quantum field evolution** into a stochastic equation
- Can also be used to treat interferometry
- Advantage: No exponential complexity issues!
- Mathematical challenge:
  - truncation error needs to be checked

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