Gaussian Phase-Space Representations IV

Vssup Lectures 2012

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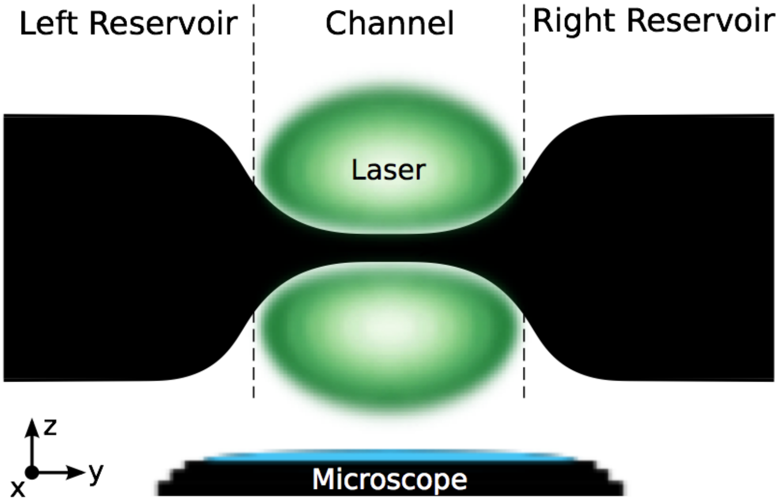
Outline

Mesoscopic cold atoms (Esslinger 2012)



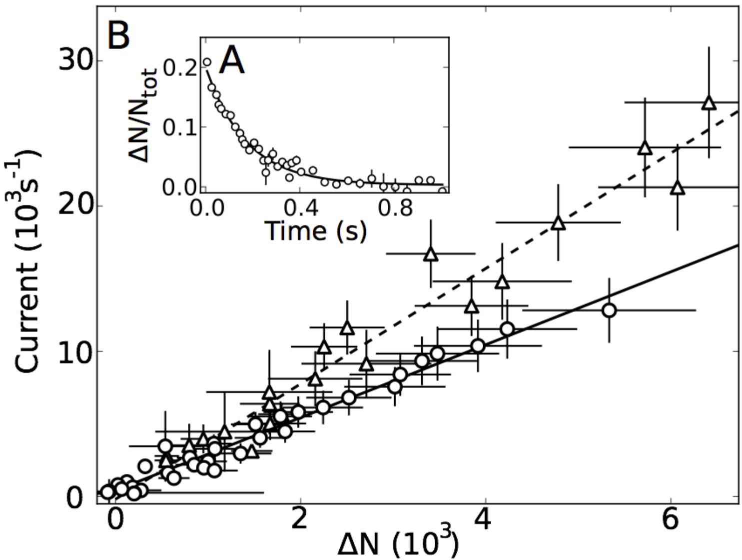
Landauer Quantum Transport

Universal quantized conductivity formula

What is the channel conductivity, ie the current in atoms per second per potential difference?

- $$G = G_0 \sum_n t_n$$
- $$G = 1/(\pi\hbar)$$
- This is a universal law found with mesoscopic electronics, **and now with mesoscopic atomtronics.**

Mesoscopic cold atoms (Esslinger 2012)



Dealing with atomic coherence in fermions

Hpw do we treat coherent phenomena with fermions?

- Is there a coherent state for fermions?
- Is there such a thing as a P-representation?
- Can we efficiently compute ground states?
- What about quantum transport?

Ways to define coherent states

- Definition 1: The coherent states $|z\rangle$ are eigenstates of the annihilation operator a: $\hat{a}|z\rangle = z|z\rangle$.
- Definition 2: The coherent states $|z\rangle$ are quantum states with a minimum uncertainty relationship: $\Delta x \Delta p = \hbar/2$
- Definition 3: The coherent states $|z\rangle$ can be obtained by applying a displacement operator $D(z)$ on the ground state of harmonic oscillator:

$$|z\rangle = D(z)|0\rangle, D(z) = exp(z\hat{a}^{\dagger} - z^{*}\hat{a})$$

General coherent states from (3)

Can we generalize coherent states?

- Consider $\mathbf{T}$ as a set of operators closed under commutation - called a **LIE ALGEBRA**
- Ie $[T_i, T_j] = \sum_k C_{ijk} T_k$
- **Define a continuous Lie group** of operators $g(\mathbf{z}) = \exp(\mathbf{T} \cdot \mathbf{z})$
- Let $|\psi_o\rangle$ be some fixed vector - the *reference* state
- Then a general coherent state is the set of states $|\mathbf{z}\rangle = \exp(\mathbf{T} \cdot \mathbf{z}) |\psi_o\rangle$
- Can get different coherent states from different $|\psi_o\rangle$.

Coherent states for fermions

- Definition 1: Gives anticommuting Grassmann variables: if $\hat{a}|z\rangle = z|z\rangle$, then a anti-commutes $\rightarrow$ z anti-commutes
- Definition 2: Not always unique, and not a complete set
- Definition 3:

Coherent states for fermions?

- Consider $|\psi_o\rangle = |1, \ldots 1, 0, \ldots 0\rangle$ **as the N-particle ground state**
- Let $|\mathbf{z}\rangle = \exp\left(\sum_{p,h} \hat{a}_p^\dagger z_{ph} \hat{a}_h\right)|\psi_o\rangle$
- For every created particle (p) we create a hole (h)

General phase-space approach

Expand density matrix in a complete basis of operators

$$\hat{\rho} = \int P(\vec{\lambda})\hat{\Lambda}(\vec{\lambda})d\vec{\lambda}$$

Phase-space may be larger still!

- Here $\hat{\Lambda}(\vec{\lambda})$ must be complete
- Quantum dynamics $\rightarrow$Trajectories in $\vec{\lambda}$.
- Different basis choice $\hat{\Lambda}(\vec{\lambda}) \rightarrow$ different representation
- Eg, positive P-representation: $\hat{\Lambda}(\vec{\lambda}) = |\alpha\rangle\langle\beta| / \langle\beta||\alpha\rangle$

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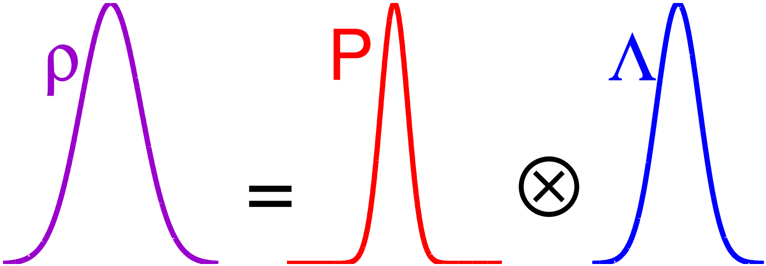
Trade-offs: distribution vs basis



$$\sigma_{\rho} \sim \sigma_{P} + \sigma_{\Lambda}$$

General Gaussian operator

General Gaussian operators give a complete basis in all cases

Normally-ordered exponential of a quadratic form in the $2M$-vector mode operator $\delta\underline{\hat{a}} = (\hat{\mathbf{a}}, \hat{\mathbf{a}}^\dagger) - \underline{\alpha}$, where $\underline{\alpha}$ is a c-vector and $\hat{\mathbf{a}}$ is the vector of annihilation operators. Used for either bosons or fermions:

$$\hat{\Lambda}(\overrightarrow{\lambda}) = \frac{\Omega}{\sqrt{\left|\underline{\underline{\sigma}}\right|}} : \exp\left[-\delta\underline{\hat{a}}^\dagger \underline{\underline{\sigma}}^{-1} \delta\underline{\hat{a}}/2\right] : \ .$$

Quantum phase-space: $\overrightarrow{\lambda} = (\Omega, \underline{\alpha}, \underline{\underline{\sigma}})$.

What is the covariance?

The covariance matrix acts as a 'stochastic Green's function'

$$\underline{\underline{\sigma}} = \begin{bmatrix} \mathbf{I}+\mathbf{n} & \mathbf{m} \\ \mathbf{m}^{+} & \mathbf{I}+\mathbf{n}^{T} \end{bmatrix}.$$

Eg, fermion case: representation phase space is $\overrightarrow{\lambda} = (\Omega, \mathbf{n}, \mathbf{m}, \mathbf{m}^{+})$

- Ω= weight factor
- $\mathbf{n}$ = number correlation - OBSERVABLE
- $\mathbf{m}, \mathbf{m}^{+}$= anomalous correlation - OBSERVABLE

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Weighted stochastic gauge equations

Exponential quantum problems → tractable stochastic equations

$$d\Omega/\partial t = \Omega\left[U + \mathbf{g}\cdot\boldsymbol{\zeta}\right]$$
$$d\boldsymbol{\alpha}/\partial t = \mathbf{A} + \mathbf{B}(\boldsymbol{\zeta} - \mathbf{g})$$

- **Can be used for fermions AND bosons**
- Can be used in imaginary time for finite temperatures
- **g is a gauge chosen to stabilize trajectories**
- A careful choice of basis, gauge and stochastic method is necessary

BOSONIC INITIAL ENSEMBLES

Nonlinear interactions at each site + linear interactions coupling different sites:

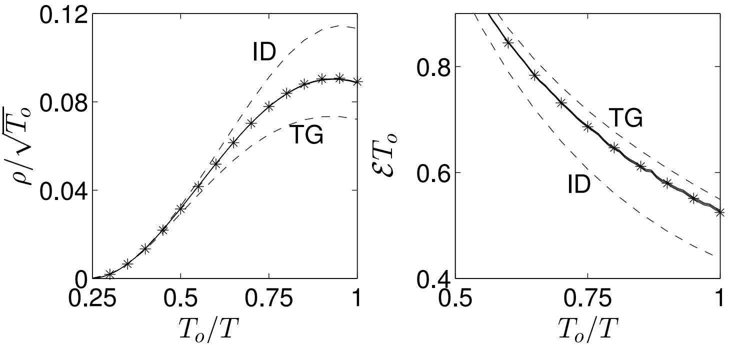
- $\widehat{H}(\mathbf{a},\mathbf{a}^{\dagger}) = \hbar \left[\sum\sum \omega_{ij} a_i^{\dagger} a_j + \sum : \widehat{n}_j^2 : \right]$.
- ω_{ij} - nonlocal coupling, includes chemical potential.
- Boson number: $\widehat{n}_i = a_i^{\dagger} a_i$.
- General approach also holds for quantum fields

A: ONE-DIMENSION, FINITE TEMPERATURE

$$\begin{aligned}
\frac{d\alpha}{d\tau} &= -\left[|\alpha\beta^*| + \omega - \nabla^2 + i\zeta_1(\tau)\right]\alpha \\
\frac{d\beta}{d\tau} &= -\left[|\alpha\beta^*| + \omega - \nabla^2 + i\zeta_2(\tau)\right]\beta \\
\frac{d\Omega}{d\tau} &= -H\Omega + \textit{gauge terms}
\end{aligned}$$

- **weighted** Gross-Pitaevskii equation + quantum noise

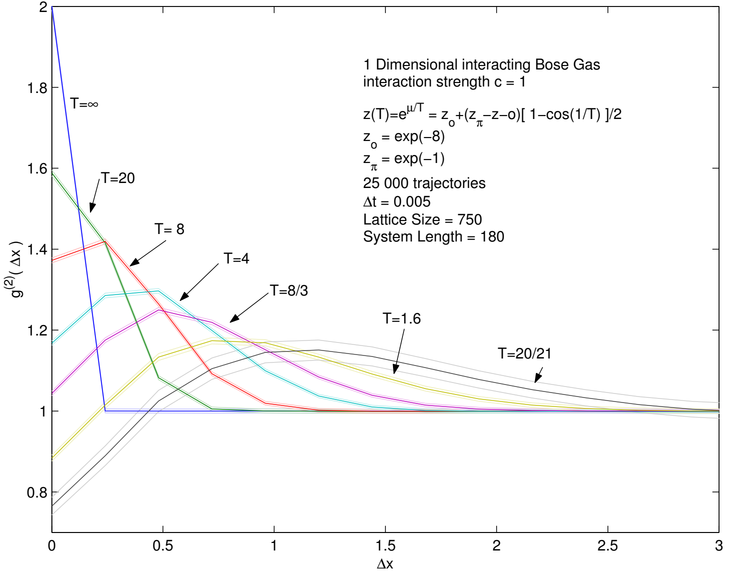
ONE-DIMENSIONAL BEC



Uses imaginary time propagation to get a finite temperature

Agreement of simulations with exact solutions

Predicts: anomalous spatial correlations



INTERACTING FERMIONS

$$\widehat{H} = -\sum_{ij,\sigma} t_{ij} \widehat{a}^{\dagger}_{i,\sigma} \widehat{a}_{j,\sigma} + U \sum_{j} : \widehat{n}_{j,j,\downarrow} \widehat{n}_{j,j,\uparrow} :$$

- Hubbard model of an interacting Fermi gas on a lattice
- Ultracold gas in an optical lattice: experiments at ETH, Zurich
  - **Weak-coupling limit $\rightarrow$ BCS transitions**
  - **Relevance to high-T_c superconductors?**
  - **Universal fermionic behavior - neutron star interiors?**

QMC sign problem

- Traditional fermionic Quantum Monte Carlo (QMC) suffers from sign problems:

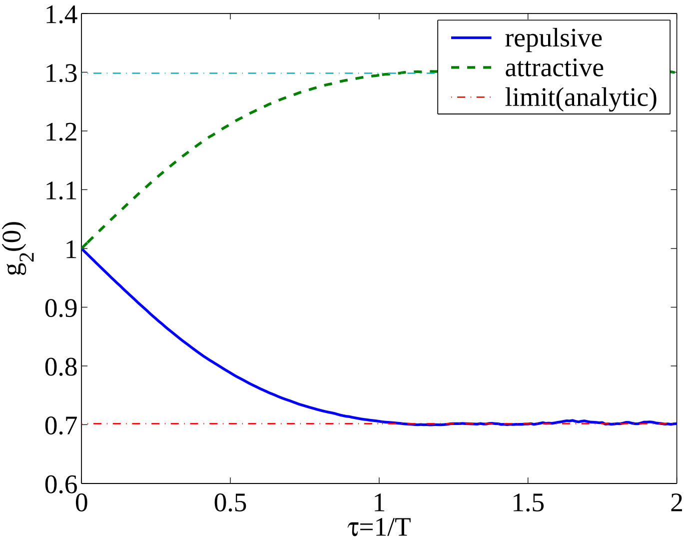
$$\langle A \rangle \sim \frac{\langle sA \rangle}{\langle s \rangle}$$

- sign problem increases with:
  - dimension,
  - lattice size,
  - interaction strength

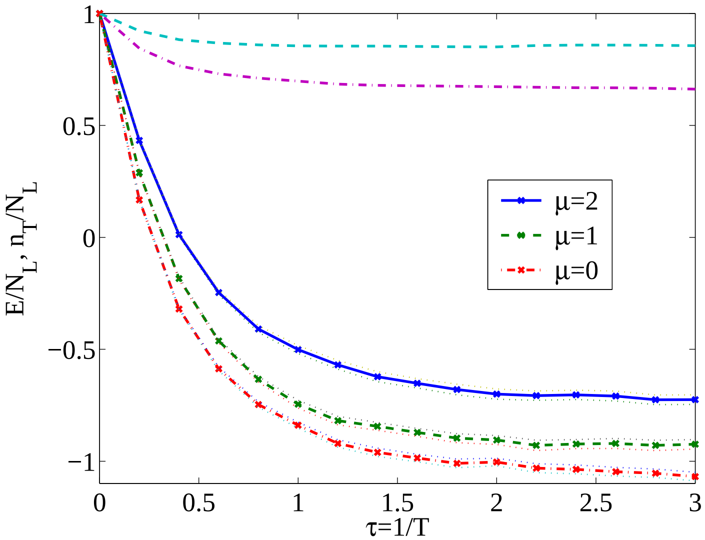
Finite-temperature phase-space equations

- Paths: $\frac{d\mathbf{n}_\sigma}{d\tau} = \frac{1}{2}\left\{\left(\mathbf{I}-\mathbf{n}_\sigma\right)T_\sigma^{(1)}\mathbf{n}_\sigma + \mathbf{n}_\sigma T_\sigma^{(2)}\left(\mathbf{I}-\mathbf{n}_\sigma\right)\right\}.$
- Weights: $\frac{d\Omega}{d\tau} = -\Omega H(\mathbf{n}_1, \mathbf{n}_{-1})$
  - T-matrix: $T_{i,j,\sigma}^{(r)} = t_{ij} - \delta_{i,j}\left\{U(n_{j,j,-\sigma} - n_{j,j,\sigma} + \frac{1}{2}) - \mu + \sigma\xi_j^{(r)}\right\}.$
  - Noises: $\left\langle \xi_j^{(r)}(\tau)\,\xi_{j'}^{(r')}(\tau')\right\rangle = 2U\delta(\tau-\tau')\delta_{j,j'}\delta_{r,r'}$.

A: 1D Lattice - 100 sites vs: exact result



B: 16x16 2D Lattice

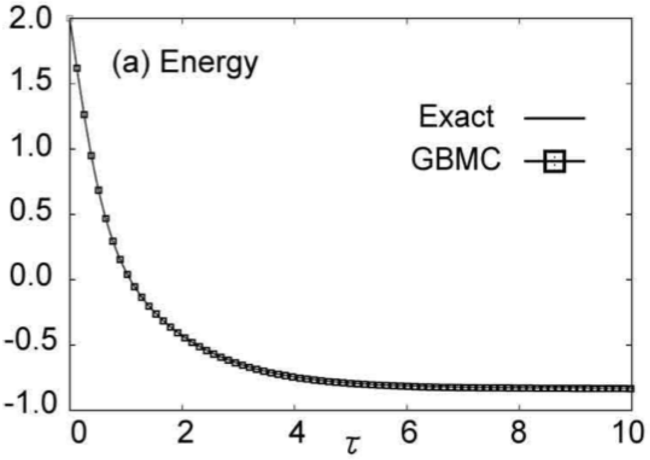


IMADA ALGORITHM

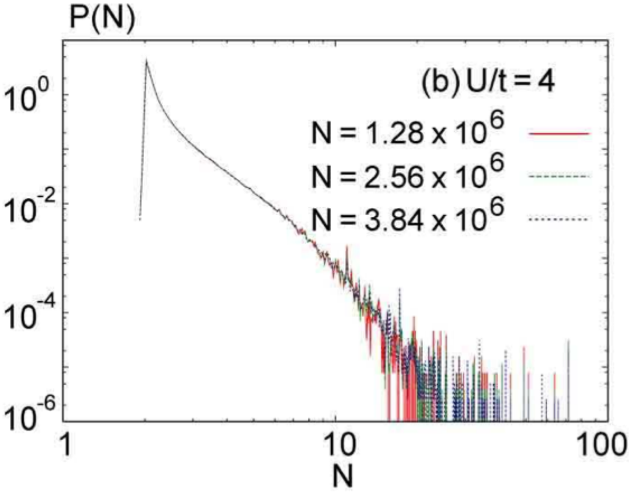
Imada **improved** on the original Gaussian method

- Used number projections to reduce the size of Hilbert space
  - Importance sampling helps to improve statistics
  - No evidence of a Fermi ‘sign’ problem
  - No evidence of boundary term problems

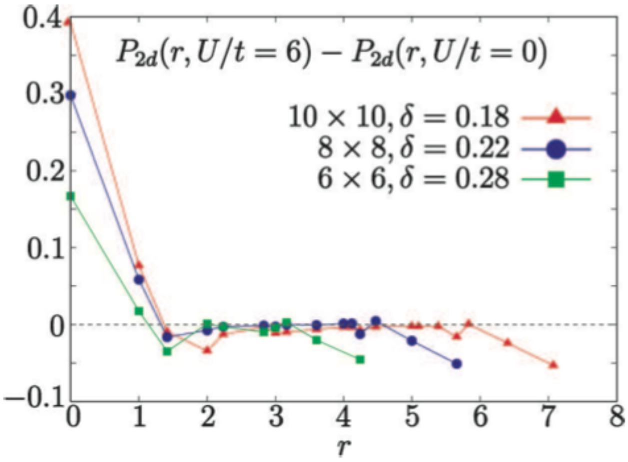
Imada algorithm test case - two sites



Imada algorithm test case - tail exponent = -4



Imada algorithm Hubbard model: no pairing



Summary

Gaussian phase-space extends to fermions

- Provides a new way to treat strongly correlated systems
  - Predicts no long-range order in Hubbard model
  - Apparently NOT the explanation of high Tc superconductors
  - To be tested in atomic Fermi gas experiments