VSSUP winter school. Basic field theory for ultracold atoms problem set

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The number operator:

(a) Use the commutation relations to show that $\hat{\psi}(\mathbf{r})$ annihilates a particles, and $\hat{\psi}^\dagger(\mathbf{r})$ creates a particle.
(b) In 1D, all N particles occupy the ground state of a harmonic trap centred around $x = 0$. If we measured the number of particles in the left half of the trap ($x < 0$), what is the variance in this measurement?
(c) Generalise your result for a measurement of the number of particles in any region $x_1 - x_2$. Express your answer in terms of $F = \int_{x_1}^{x_2} |u_0(x)|^2 \, dx$, where $u_0(x)$ is the single particle ground state.
(d) Generalise the result from the previous question to a state with number variance V_0 in the ground state and vacuum in all the excited modes.

Dynamics:

(a) The many-body Hamiltonian for a system of identical bosons is

$$\mathcal{H} = \int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r}) \hat{H} \hat{\psi}(\mathbf{r}) \; d^3\mathbf{r} + \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}^\dagger(\mathbf{r}') U(\mathbf{r} - \mathbf{r}') \hat{\psi}(\mathbf{r}) \hat{\psi}(\mathbf{r}') \; d^3\mathbf{r} \, d^3\mathbf{r}' \tag{1}$$

(b) The number operator is $\hat{N} = \int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r}) \hat{\psi}(\mathbf{r}) \; d^3\mathbf{r}$. Show that the number of particles is a conserved quantity.
(c) Starting from the many-body hamiltonian, derive the equation of motion for $\hat{\psi}(\mathbf{r})$.
(d) Consider a harmonic potential in 1 dimension $V(x) = \frac{1}{2} m\omega^2 x^2$, where m is the mass of the particle. For low speeds, the *single-particle* Hamiltonian is then $H = \frac{\hat{p}^2}{2m} + V(x)$. It turns out that the eigenvalues of this Hamiltonian are evenly spaced: $H u_k(x) = E_k u_k(x) \equiv \hbar\omega(k + \frac{1}{2}) u_k(x)$, where k is an integer ranging from 0 to ∞. Assume there are many non interacting ($U_0 = 0$) identical bosons confined in this potential.
Calculate $\langle \hat{x} \rangle$ as a function t for the same potential, if

$$|\Psi(t = 0)\rangle = \frac{1}{\sqrt{2}} \left(|N, N-1, 0, 0, \ldots\rangle - i |N-1, N, 0, 0, \ldots\rangle \right) \tag{2}$$

Hint: You might find it useful to notice that for the special case of the harmonic oscillator, the single particle position operator can be written as $x = \sqrt{\frac{\hbar}{2m\omega}} \left(\hat{b} + \hat{b}^\dagger \right)$, where $\hat{b}^\dagger$ and $\hat{b}$ are the *single-particle* harmonic oscillator raising and lowering operators, which have the properties

$$\left[\hat{b} \, , \, \hat{b}^\dagger \right] = 1 \, , \quad \hat{b} u_k(x) = \sqrt{k} u_{k-1}(x) \, , \quad \hat{b}^\dagger u_k(x) = \sqrt{k+1} u_{k+1}(x) \tag{3}$$

Please, please, do not get these confused with the many particle creation and annihilation operators.

Atom Interferometry:

The goal of this section is to derive the standard quantum limit for atom interferometry. An atom interferometer can be formed from an ensemble of two-level atoms with 2 non-degenerate ground states coupled by a microwave radiation field. In lectures, we showed that the Hamiltonian for a system of atoms interacting with a radiation field is

$$\begin{aligned}\mathcal{H} &= \int_{-\infty}^{\infty} \hat{\psi}_1^\dagger(\mathbf{r}) H_0 \hat{\psi}_1(\mathbf{r})\ d^3\mathbf{r} + \int_{-\infty}^{\infty} \hat{\psi}_2^\dagger(\mathbf{r})(H_0 + \hbar\omega)\hat{\psi}_2(\mathbf{r})\ d^3\mathbf{r} \\ &+ \hbar\Omega \int_{-\infty}^{\infty} \left(\hat{\psi}_1(\mathbf{r})\hat{\psi}_2^\dagger(\mathbf{r})e^{-i\omega t} + \hat{\psi}_1^\dagger(\mathbf{r})\hat{\psi}_1^\dagger(\mathbf{r})e^{i\omega t}\right) dr \qquad (4)\end{aligned}$$

step 1

Assuming that the spatial dynamics is unimportant, we can approximate the system as a two-mode system. Show that if we make the substitution $\hat{\psi}_1(\mathbf{r}) \to \hat{a}u_0(\mathbf{r})$, $\hat{\psi}_2(\mathbf{r}) \to \hat{b}u_0(\mathbf{r})$, where $H_0 u_0(\mathbf{r}) = \hbar\omega_0 u_0$, and $[\hat{a}, \hat{a}^\dagger] = [\hat{b}, \hat{b}^\dagger] = 1$, show that we can now write the Hamiltonian in a more simple form:

$$\mathcal{H} = \hbar\omega_0(\hat{a}^\dagger\hat{a} + \hat{b}^\dagger\hat{b}) + \hbar\omega\hat{b}^\dagger\hat{b} + \hbar\Omega\left(\hat{a}\hat{b}^\dagger e^{-i\omega t} + \hat{b}\hat{a}^\dagger e^{i\omega t}\right) \qquad (5)$$

step 2

Show that the Heisenberg equations of motion are

$$\begin{aligned} i\frac{d}{dt}\hat{a} &= \omega_0\hat{a} + \Omega\hat{b}e^{i\omega t} \qquad (6)\\ i\frac{d}{dt}\hat{b} &= (\omega_0 + \omega)\hat{b} + \Omega\hat{a}e^{-i\omega t} \qquad (7)\end{aligned}$$

step 3

Show that by making the transformation $\tilde{a} = \hat{a}e^{i\omega_0 t}$ and $\tilde{b} = \hat{b}e^{i(\omega+\omega_0)t}$, the equations of motion simplify to

$$\begin{aligned} i\frac{d}{dt}\tilde{a} &= \Omega\tilde{b} \qquad (8)\\ i\frac{d}{dt}\tilde{b} &= \Omega\tilde{a} \qquad (9)\end{aligned}$$

step 4

show that the solution to these equations is

$$\begin{aligned} \tilde{a}(t) &= \tilde{a}_0\cos\Omega t - i\tilde{b}_0\sin\Omega t \qquad (10)\\ \tilde{b}(t) &= \tilde{b}_0\cos\Omega t - i\tilde{a}_0\sin\Omega t \qquad (11)\end{aligned}$$

step 5

After an amount of time $t_1 = \frac{\pi}{4\Omega}$, we set $\Omega = 0$, such that

$$\begin{aligned} \tilde{a}(t_1) &= \frac{1}{\sqrt{2}}\left(\tilde{a}_0 - i\tilde{b}_0\right) \qquad (12)\\ \tilde{b}(t_1) &= \frac{1}{\sqrt{2}}\left(\tilde{b}_0 - i\tilde{a}_0\right). \qquad (13)\end{aligned}$$

After evolving under the Hamiltonian $\mathcal{H}_1 = \hbar\omega_p \hat{b}^\dagger \hat{b}$ for an amount of time Δt, mode b acquires a phase shift $\phi \equiv \omega_p \Delta t$, such that

$$\tilde{a}(t_2) = \frac{1}{\sqrt{2}}\left(\tilde{a}_0 - i\tilde{b}_0\right) \tag{14}$$

$$\tilde{b}(t_2) = \frac{1}{\sqrt{2}}\left(\tilde{b}_0 - i\tilde{a}_0\right)e^{i\phi}\,. \tag{15}$$

Finally, we switch Ω back on for an amount of time $t = \frac{\pi}{4\Omega}$, such that

$$\tilde{a}(t_3) = \frac{1}{2}(1 - e^{i\phi})\hat{a}_0 - \frac{i}{2}(1 + e^{i\phi})\hat{b}_0 \tag{16}$$

$$\tilde{b}(t_3) = \frac{1}{2}(-1 + e^{i\phi})\hat{b}_0 - \frac{i}{2}(1 + e^{i\phi})\hat{a}_0\,. \tag{17}$$

Assuming the initial state $|\Psi\rangle = |N, 0\rangle$, calculate $S \equiv \langle(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})\rangle$ at $t = t_3$.

step 6

Assuming the initial state $|\Psi\rangle = |N, 0\rangle$, calculate the variance in $(\hat{a}^\dagger\hat{a} - \hat{b}^\dagger\hat{b})$, and find the phase uncertainty $\Delta\phi \equiv \sqrt{\frac{V(S)}{(\frac{dS}{d\phi})^2}}$.