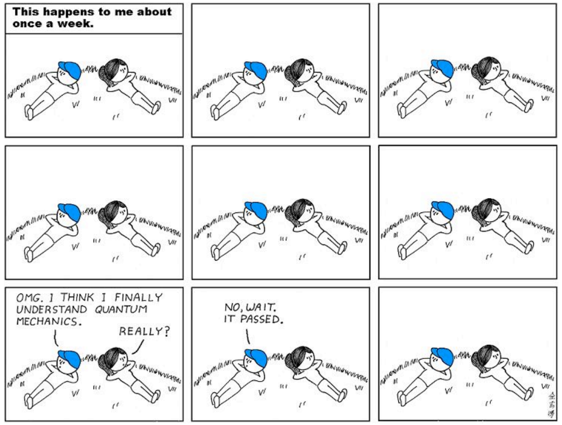
Introduction to quantum field theory for ultra-cold atoms



Simon Haine
University of Queensland

outline: nothing is certain

- QM with many particles
- 2nd quantisation
- ultra-cold atoms
- scattering
- mean field theory
- Quantisation of the EM field
- Interaction of atoms and light
- Fun examples
  - the atom laser
  - atom interferometry

Lecture 1

Lecture 2

Lecture 3

Revision: Basic postulates of Quantum mechanics:

(1) The system is represented by a quantum state $|\Psi(t)\rangle$

(2) Observables quantities are represented by Hermitian operators.

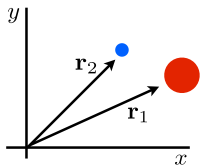
(3) If you make a measurement of an observable, you are certain to get one of the eigenvalues of that operator.

(4) The probability of getting this value is equal to the mod-square of the projection of the state along that corresponding eigenstate.

$$\hat{Q}|\phi_n\rangle = \lambda_n|\phi_n\rangle$$

$$|\Psi(t)\rangle = \sum_n c_n(t)|\phi_n\rangle \implies P(\lambda_n) = |c_n|^2 = |\langle\phi_n|\Psi_n(t)\rangle|^2$$

Clicker question:



When we describe this system quantum mechanically, in general, the wavefunction for the system $\Psi(\mathbf{r}_1, \mathbf{r}_2)$ will be of the form:

(a) $f(\mathbf{r}_1) + g(\mathbf{r}_2)$

(b) $f(\mathbf{r}_1)g(\mathbf{r}_2)$

(c) $\sum_j b_j f_j(\mathbf{r}_1) \sum_k c_k g_k(\mathbf{r}_2)$

(d) $\sum_j \sum_k c_{jk} f_j(\mathbf{r}_1) g_k(\mathbf{r}_2)$

(e) $\sum_j f_j(\mathbf{r}_1 \times \mathbf{r}_2)$

(f) None of the above

Many-body Quantum Mechanics

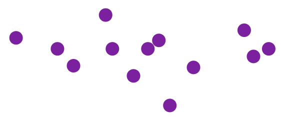
$$\Psi(r_1, r_2, \ldots, r_n, t) = \psi_1(r_1, t)\psi_2(r_2, t)\ldots\psi_n(r_n, t)$$

(special case where there is absolutely no entanglement between the particles, and they stay that way)

$$\Psi(r_1, r_2, \ldots, r_n, t) = \sum_{j_1}\sum_{j_2}\cdots\sum_{j_n} c_{j_1, j_2, \ldots, j_n}(t)\psi_{j_1}(r_1)\psi_{j_2}(r_2)\ldots\psi_{j_n}(r_n)$$

(The unfortunate reality)

Identical (indistinguishable) particles:



What does it mean for particles to be indistinguishable?

Identical particles:

Define exchange operator:

$$\hat{P}_{ij}\Psi(r_1, \ldots, r_i, \ldots, r_j, \ldots r_n) = \Psi(r_1, \ldots, r_j, \ldots, r_i, \ldots r_n)$$

If particles are indistinguishable, then physics can't change by swapping two particles around, so:

$$\hat{P}_{ij}\Psi(r_1, r_2, \ldots, r_n) = e^{i\phi}\Psi(r_1, r_2, \ldots, r_n)$$

And, obviously: $\hat{P}_{ij}^2\Psi(r_1, \ldots, r_n) = \Psi(r_1, \ldots, r_n)$

So, two choices: $\hat{P}_{ij}\Psi(r_1, \ldots, r_n) = \pm\Psi(r_1, \ldots, r_n)$

We call '+' case 'bosons' and '-' case 'fermions'

Spin statistics theorem (from relativistic QM) says particles with:
integer spin are bosons, **half integer** spin are fermions.

Eg:

Bosons:

$$\Psi(r_1, r_2) = \frac{1}{2}(\psi_1(r_1)\psi_2(r_2) + \psi_2(r_1)\psi_1(r_2))$$

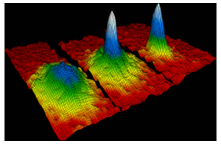
Fermions:

$$\Psi(r_1, r_2) = \frac{1}{2}(\psi_1(r_1)\psi_2(r_2) - \psi_2(r_1)\psi_1(r_2))$$

Eg:

Bosons:

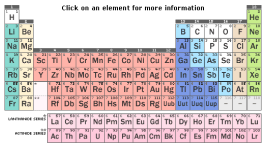
$$\Psi(r_1,r_2) = \frac{1}{2}(\psi_1(r_1)\psi_1(r_2) + \psi_1(r_1)\psi_1(r_2))$$
$$= \psi_1(r_1)\psi_1(r_2) \longrightarrow \text{BEC!}$$



Fermions:

$$\Psi(r_1,r_2) = \frac{1}{2}(\psi_1(r_1)\psi_1(r_2) - \psi_1(r_1)\psi_1(r_2))$$
$$= 0$$

Unphysical → Pauli exclusion principle: No two fermions can have the same quantum numbers



→ Chemistry

Fermions:

- Electrons (atomic structure, metals, semi-conductors)
- protons, neutrons, quarks, leptons
- in general, matter is made of fermions

Bosons:

- Photons
- 'Force carriers' (photon is the force carrier for EM force, Gluons for the strong force, W+ etc for the week force. Higgs boson. Gravitons?)
- Collections of fermionic particles can behave like bosons, eg: atoms with an even number of neutrons, quasi-particles (eg, phonons, cooper pairs).

Clicker Question:

Is this state symmetric?

$$\Psi(r_1, r_2, \ldots, r_n, t) = \psi_1(r_1,t)\psi_2(r_2,t)\ldots\psi_n(r_n,t)$$

(1) Yes

(2) No

(3) it depends.

N Identical Bosons:

Symeterized version of this state:

$$\Psi(r_1, r_2, \ldots, r_n, t) = \psi_1(r_1,t)\psi_2(r_2,t)\ldots\psi_n(r_n,t)$$

is:
$$\Psi(r_1, r_2, \ldots, r_n, t) = \frac{1}{n!}\big(\psi_1(r_1,t)\psi_2(r_2,t)\ldots\psi_n(r_n,t)$$
$$+\psi_2(r_1,t)\psi_1(r_2,t)\ldots\psi_n(r_n,t)$$
$$+\psi_n(r_1,t)\psi_1(r_2,t)\ldots\psi_2(r_n,t)$$
$$+\text{all other permutations}\big)$$

Easier way: 2nd quantisation (sometimes called quantum field theory)

Define basis ket

$$|n_1, n_2, \ldots, n_k\rangle \equiv \sqrt{\frac{N!}{n_1! n_2! \ldots n_k!}} \Big(\underbrace{\psi_1 \ldots \psi_1}_{n_1 \text{times}} \underbrace{\psi_2 \ldots \psi_2}_{n_2 \text{times}} \cdots \underbrace{\psi_k \ldots \psi_k}_{n_k \text{times}}$$

$$+\psi_2 \underbrace{\psi_1 \ldots \psi_1}_{n_1 \text{times}} \underbrace{\psi_2 \ldots \psi_2}_{n_2 - 1 \text{times}} \cdots \underbrace{\psi_k \ldots \psi_k}_{n_k \text{times}}$$

$$+\text{all other permutations}\Big)$$

$$|n_1, n_2, \ldots, n_k\rangle$$

- is called a 'Fock state' or 'number state'
- has n_1 atoms in mode 1, n_2 atoms in mode 2 etc.
- General state is then:

$$|\Psi\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} C_{n_1, n_2, \ldots, n_k}(t) |n_1, n_2, \ldots, n_k\rangle$$

Examples:

$$\frac{1}{\sqrt{2}} (\psi_2 \psi_1 + \psi_1 \psi_2) =$$

$$\psi_1 \psi_1 =$$

$$\frac{1}{\sqrt{6}} (\psi_1 \psi_2 \psi_3 + \psi_1 \psi_3 \psi_2 + \psi_2 \psi_1 \psi_3 + \psi_3 \psi_1 \psi_2 + \psi_2 \psi_3 \psi_1 + \psi_3 \psi_2 \psi_1)$$

$$=$$

Other useful properties:

Orthogonal:

$$\langle n_1, n_2, \ldots, n_k | n_1', n_2', \ldots, n_k' \rangle = \delta_{n_1, n_1'} \delta_{n_2, n_2'} \ldots \delta_{n_k, n_k'}$$

We call:

$$|0, 0, 0, \ldots, 0\rangle$$

The *vacuum* ket

- Imagine N particles. Classically, you'd need to know the position and momentum of each particle. Thats $6N$ numbers.

Quantum:

$$|\Psi\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_k=0}^{\infty} C_{n_1,n_2,\ldots,n_k}(t)|n_1,n_2,\ldots,n_k\rangle$$

- N^k complex numbers needed to describe N particles in k modes!

- For 10^6 particles in 100 modes, thats $(10^6)^{100} = 10^{600}$

Clicker Question:

Can all many-body wavefunctions $\Psi(r_1, r_2, \ldots, r_n)$ be represented as a superposition of Fock states $|\Psi\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_k=0}^{\infty} C_{n_1,n_2,\ldots,n_k}(t)|n_1,n_2,\ldots,n_k\rangle$?

- Yes

- No

- Not enough info.

Clicker Question:

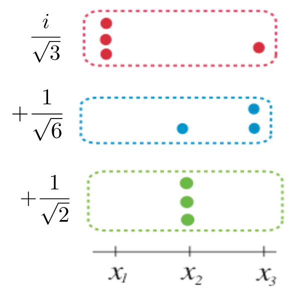
Can all superposition of Fock states $|\Psi\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_k=0}^{\infty} C_{n_1,n_2,\ldots,n_k}(t)|n_1,n_2,\ldots,n_k\rangle$ be represented as a many-body wavefunctions $\Psi(r_1, r_2, \ldots, r_n)$?

- Yes

- No

- Not enough info.

Example:



$$|\Psi\rangle = \frac{1}{\sqrt{2}}|0,3,0\rangle + \frac{1}{\sqrt{6}}|0,1,2\rangle + \frac{i}{\sqrt{3}}|3,0,1\rangle$$

Clicker Question:

$$|\Psi\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_k=0}^{\infty} C_{n_1,n_2,\ldots,n_k}(t)|n_1,n_2,\ldots,n_k\rangle$$

Are the coeffecients (or mod squared of them $|C_{n_1,n_2,\ldots,n_k}|^2$) conserved?

(1) Yes

(2) No

(3) Depends

(4) In general, we can't say

How do we calculate stuff with these states?

Eg:

$$|\Psi\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_k=0}^{\infty} C_{n_1,n_2,\ldots,n_k}(t)|n_1,n_2,\ldots,n_k\rangle$$

What is <x> ?

Creation and annihilation operators:

$$\left[\hat{a}_i\,,\ \hat{a}_j^\dagger\right] = \delta_{ij}$$

$$[\hat{a}_i\,,\ \hat{a}_j] = \left[\hat{a}_i^\dagger\,,\ \hat{a}_j^\dagger\right] = 0$$

$$\hat{a}_j|n_1,\ldots,n_j,\ldots\rangle = \sqrt{n_j}|n_1,\ldots,n_j-1,\ldots\rangle$$

$$\hat{a}_j^\dagger|n_1,\ldots,n_j,\ldots\rangle = \sqrt{n_j+1}|n_1,\ldots,n_j+1,\ldots\rangle$$

Number operators:

$$\hat{a}_j^\dagger\hat{a}_j|n_1,\ldots,n_j,\ldots\rangle = n_j|n_1,\ldots,n_j,\ldots\rangle$$

$$\hat{N}_j = \hat{a}_j^\dagger\hat{a}_j$$

The Fock states are eigenstates of $\hat{N}_j$ with eigenvalue n_j.

Total number of particles is given by the operator $\hat{N} = \sum_j \hat{N}_j = \sum_j \hat{a}_j^\dagger\hat{a}_j$

Some people use the creation operator as a way of generating the number states:

$$|0,\ldots,n_j,\ldots,0\rangle = \frac{(\hat{a}_j^\dagger)^{n_j}}{\sqrt{n_j!}}|0,\ldots,0,\ldots,0\rangle$$

And you can keep doing this:

$$|n_1,n_2\ldots,n_j,\ldots\rangle = \frac{1}{\sqrt{n_1!}\sqrt{n_2!}\ldots\sqrt{n_j!}}(\hat{a}_1^\dagger)^{n_1}(\hat{a}_2^\dagger)^{n_2}\ldots(\hat{a}_j^\dagger)^{n_j}|0,0,\ldots,0,\ldots\rangle$$

Clicker Question:

$|\Psi\rangle = |3,0,12,17,0\rangle$ What is $\langle \hat{a}_3 \hat{a}_3^\dagger \rangle$?

(a) 0

(b) 1

(c) 2

(d) 3

(e) 11

(f) 12

(g) 13

(h) 17

(i) not enough info

Fermions:

$$\{\hat{a}_i\,,\,\hat{a}_j^\dagger\} = \delta_{ij} \qquad \{\hat{a}_i\,,\,\hat{a}_j\} = \{\hat{a}_i^\dagger\,,\,\hat{a}_j^\dagger\} = 0$$

$$\{\hat{A}\,,\,\hat{B}\} \equiv \hat{A}\hat{B} + \hat{B}\hat{A}$$

$$|0,\ldots,n_j,\ldots,0\rangle = \frac{(\hat{a}_j^\dagger)^{n_j}}{\sqrt{n_j!}}|0,\ldots,0,\ldots,0\rangle$$

$$\hat{a}_j\hat{a}_j + \hat{a}_j\hat{a}_j = 0 \implies \hat{a}_j^2 = 0$$

$$\hat{a}_j^\dagger\hat{a}_j^\dagger + \hat{a}_j^\dagger\hat{a}_j^\dagger = 0 \implies (\hat{a}_j^\dagger)^2 = 0$$

ensures you never get more than one fermion in each mode. Its pretty easy to show:

$$\hat{a}^\dagger|0\rangle = |1\rangle \qquad \hat{a}|1\rangle = |0\rangle$$

$$\hat{a}^\dagger|1\rangle = 0 \qquad \hat{a}|0\rangle = 0$$

need to be careful with the sign:

$$\hat{a}_i \hat{a}_j = -\hat{a}_j \hat{a}_i \qquad \hat{a}_i \hat{a}_j^\dagger = -\hat{a}_j^\dagger \hat{a}_i$$

$$\hat{a}_j |n_1, \ldots, n_j, \ldots\rangle$$

$$= \hat{a}_j (\hat{a}_1^\dagger)^{n_1} (\hat{a}_2^\dagger)^{n_2} \ldots (\hat{a}_j^\dagger)^{n_j} \ldots |0\rangle$$

$$= (-1)^{J_j} (\hat{a}_1^\dagger)^{n_1} (\hat{a}_2^\dagger)^{n_2} \ldots (\hat{a}\hat{a}_j^\dagger) \ldots |0\rangle \qquad J_j \equiv n_1 + n_2 + \ldots n_{j-1}$$

so:

$$\hat{a}_j |\ldots n_j \ldots\rangle = (-1)^{J_j} \sqrt{n_j} |\ldots n_j - 1, \ldots\rangle \qquad \text{if } n_j = 1, \text{ 0 otherwise}$$

$$\hat{a}_j^\dagger |\ldots n_j \ldots\rangle = (-1)^{J_j} \sqrt{n_j + 1} |\ldots n_j + 1, \ldots\rangle \qquad \text{if } n_j = 0, \text{ 0 otherwise}$$

$$\hat{a}_j^\dagger \hat{a}_j |\ldots n_j \ldots\rangle = n_j |\ldots n_j, \ldots\rangle \qquad n_j = 0, 1$$

29

Dynamics:

Many body wavefunction obeys the Schrodinger equation:

$$i\hbar \frac{d}{dt} \Psi(r_1, r_2, \ldots, r_n) = \hat{\mathcal{H}} \Psi(r_1, r_2, \ldots, r_n)$$

$\hat{\mathcal{H}}$ is just the sum of the Kinetic energy and external potential energys for the individual particles, and the interparticle interactions:

$$\hat{\mathcal{H}} = \sum_j \hat{H}_j + \frac{1}{2} \sum_{i,j} U(r_i - r_j)$$

$$= \sum_j \left(\frac{-\hbar^2}{2m} \nabla_j^2 + V_{ex}(r_j) \right) + \frac{1}{2} \sum_{i,j} U(r_i - r_j)$$

30

1st quantised representation: Expand in a basis

$$\hat{H}_j = \sum_f \sum_g |f\rangle\langle f| \, \hat{H}_j \, |g\rangle\langle g|$$

$$= \sum_f \sum_g H_{f,g} |f\rangle\langle g| \qquad H_{fg} = \langle f|\hat{H}_j|g\rangle$$

$$\hat{\mathcal{H}} = \sum_j \sum_{f,g} H_{fg} |f_j\rangle\langle g_j| + \sum_{i,j} \sum_{e,f,g,h} U_{e,f,g,h} |e_i\rangle|f_j\rangle\langle g_i|\langle h_j|$$

Why do we end up with 2 projectors here?

31

2nd quantised representation of Hamiltonian:

$$\hat{\mathcal{H}} = \sum_j \sum_{f,g} H_{fg} |f_j\rangle\langle g_j| + \sum_{i,j} \sum_{e,f,g,h} U_{e,f,g,h} |e_i\rangle|f_j\rangle\langle g_i|\langle h_j|$$

Swap the order of the summation:

$$\hat{\mathcal{H}} = \sum_{f,g} H_{fg} \sum_j \Big(|f_j\rangle\langle g_j| \Big) + \sum_{e,f,g,h} U_{e,f,g,h} \sum_{i,j} \Big(|e_i\rangle|f_j\rangle\langle g_i|\langle h_j| \Big)$$

A straightforward calculation with far far too many indices shows that operators in brackets can be written very easily with our creation and annihilation operators:

$$\hat{\mathcal{H}} = \sum_{f,g} H_{fg} \hat{a}_f^\dagger \hat{a}_g + \sum_{e,f,g,h} U_{efgh} \hat{a}_e^\dagger \hat{a}_f^\dagger \hat{a}_g \hat{a}_h$$

32

2nd quantised representation of Hamiltonian:

$$\hat{\mathcal{H}} = \sum_{f,g} H_{fg} \hat{a}_f^\dagger \hat{a}_g + \sum_{e,f,g,h} U_{efgh} \hat{a}_e^\dagger \hat{a}_f^\dagger \hat{a}_g \hat{a}_h$$

$$H_{fg} = \int_{-\infty}^{\infty} \psi_f^*(\mathbf{r}) \hat{H} \psi_g(\mathbf{r})\, d^3\mathbf{r}$$

$$U_{efgh} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_e^*(\mathbf{r}_1) \psi_f^*(\mathbf{r}_2) U(\mathbf{r}_1 - \mathbf{r}_2) \psi_g(\mathbf{r}_1) \psi_f(\mathbf{r}_2)\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2$$

Even easier if you pick the right basis

(ie, $\psi_f(\mathbf{r})$ are eigenstates of $\hat{H}$)

$$\hat{\mathcal{H}} = \sum_{f} E_f \hat{a}_f^\dagger \hat{a}_f + \sum_{e,f,g,h} U_{efgh} \hat{a}_e^\dagger \hat{a}_f^\dagger \hat{a}_g \hat{a}_h$$

$$\hat{H} \psi_f(\mathbf{r}) = E_f \psi_f(\mathbf{r})$$

$$U_{efgh} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \psi_e^*(\mathbf{r}_1) \psi_f^*(\mathbf{r}_2) U(\mathbf{r}_1 - \mathbf{r}_2) \psi_g(\mathbf{r}_1) \psi_f(\mathbf{r}_2)\, d^3\mathbf{r}_1\, d^3\mathbf{r}_2$$

No interactions:

$$\hat{\mathcal{H}} = \sum_j E_j \hat{a}_j^\dagger \hat{a}_j = \sum_j E_j \hat{N}_j$$

quantum optics, for example

The field operator:

$$\hat{\psi}(\mathbf{r}) = \sum_j \hat{a}_j u_j(\mathbf{r})$$

$$\hat{\psi}^\dagger(\mathbf{r}) = \sum_j \hat{a}_j^\dagger u_j^*(\mathbf{r})$$

$$\left[\hat{\psi}(\mathbf{r})\,,\, \hat{\psi}^\dagger(\mathbf{r}')\right] = \delta(\mathbf{r} - \mathbf{r}')$$

$$\left[\hat{\psi}(\mathbf{r})\,,\, \hat{\psi}(\mathbf{r}')\right] = \left[\hat{\psi}^\dagger(\mathbf{r})\,,\, \hat{\psi}^\dagger(\mathbf{r}')\right] = 0$$

Fermions

$$\hat{\psi}(\mathbf{r}) = \sum_j \hat{a}_j u_j(\mathbf{r})$$

$$\hat{\psi}^\dagger(\mathbf{r}) = \sum_j \hat{a}_j^\dagger u_j^*(\mathbf{r})$$

$$\{\hat{\psi}(\mathbf{r})\,,\,\hat{\psi}^\dagger(\mathbf{r}')\} = \delta(\mathbf{r}-\mathbf{r}')$$

$$\{\hat{\psi}(\mathbf{r})\,,\,\hat{\psi}(\mathbf{r}')\} = \{\hat{\psi}^\dagger(\mathbf{r})\,,\,\hat{\psi}^\dagger(\mathbf{r}')\} = 0$$

From now on, assume i'm talking about bosons, unless I specifically say otherwise. But it all generalises to fermions in a fairly obvious way

$$\hat{\mathcal{H}} = \sum_{i,j} H_{i,j}\hat{a}_i^\dagger \hat{a}_j \qquad H_{ij} = \int_{-\infty}^{\infty} u_i^*(\mathbf{r})\hat{H}u_j(\mathbf{r})\,d^3\mathbf{r}$$

$$= \sum_i \sum_j \int_{-\infty}^{\infty} u_i^*(\mathbf{r})Hu_j(\mathbf{r})\;d^3\mathbf{r}\hat{a}_i^\dagger\hat{a}_j$$

$$= \int_{-\infty}^{\infty} \sum_i \hat{a}_i^\dagger u_i^*(\mathbf{r})H\sum_j u_j(\mathbf{r})\hat{a}_j\;d^3\mathbf{r}$$

$$= \int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})\hat{H}\hat{\psi}(\mathbf{r})\;d^3\mathbf{r} \qquad \hat{\psi}(\mathbf{r}) = \sum_j \hat{a}_j u_j(\mathbf{r})$$

Hamiltonian:

$$\hat{\mathcal{H}} = \int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})\hat{H}\hat{\psi}(\mathbf{r})\;d^3\mathbf{r} + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')U(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}')\;d^3\mathbf{r}\,d^3\mathbf{r}'$$

This same process actually works for any one-body operator. Eg, the position operator that operators on our many-body state is now:

$$\hat{x} = \int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})x\hat{\psi}(\mathbf{r})\;d^3\mathbf{r}$$

So the 'average' position of all our particles is

$$\langle\hat{x}\rangle = \langle\Psi|\int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})x\hat{\psi}(\mathbf{r})\;d^3\mathbf{r}|\Psi\rangle$$

Number operator

$$\hat{N} = \int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\ d^3\mathbf{r}$$

Expectation value of the density:

$$\langle n(x)\rangle = \langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\rangle$$

Clicker Question:

At t=0, we have n particles in the ground state of an infinite square well. ie

$$|\Psi\rangle = |N, 0, 0, \ldots, 0\rangle$$

We then instantaneously turn our potential into a harmonic oscillator. If we measured the variance of the number of particles in the (new) ground state, what would we find?

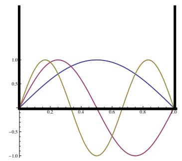
(1) 0

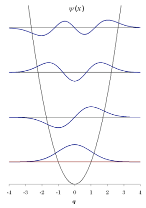
(2) something nonzero

(3) not enough info

Say our state is $|\Psi\rangle = |N, 0, 0, \ldots, 0\rangle$

How do we change basis?





Say our state is $|\Psi\rangle = |N, 0, 0, \ldots, 0\rangle$

How do we change basis?

In general, our states in the new basis will look something like

$$|\Psi\rangle = \sum_{n_1, n_2, \ldots, n_k} c_{n_1, n_2, \ldots, n_k} |n_1, n_2, \ldots, n_k\rangle$$

So we'll need some **HUGE** matrix to transform all the coeffecients!

Easier way:
Just change the basis of your operator!

$$|\Psi\rangle = |N, 0, 0, \ldots, 0\rangle$$

$$\hat{\psi}(\mathbf{r}) = \sum_j \hat{a}_j u_j(\mathbf{r}) = \sum_j \hat{b}_j v_j(\mathbf{r})$$

$$\hat{b}_i = \sum_j A_{ij}\hat{a}_j \qquad A_{ij} = \int_{-\infty}^{\infty} v_i^*(\mathbf{r}) u_j(\mathbf{r})\; d^3\mathbf{r}$$

Example:

$$|\Psi\rangle = |N_0, N_1, 0, \ldots, 0\rangle \qquad \hat{\psi}(\mathbf{r}) = \sum_j \hat{a}_j u_j(\mathbf{r}) \;= \sum_j \hat{b}_j v_j(\mathbf{r})$$

number of particles initially in ground state of square well

square well basis states

H.O basis states

Whats the expectation number of particles in the ground state of the **SQUARE WELL**?

$$\langle \hat{a}_0^\dagger \hat{a}_0\rangle = \langle N_0, N_1, 0, \ldots |\hat{a}_0^\dagger \hat{a}_0| N_0, N_1, 0, \ldots\rangle = N_0$$

(easy)

Example:

$$|\Psi\rangle = |N_0, N_1, 0, \ldots, 0\rangle \qquad \hat{\psi}(\mathbf{r}) = \sum_j \hat{a}_j u_j(\mathbf{r}) \;= \sum_j \hat{b}_j v_j(\mathbf{r})$$

number of particles initially in ground state of harmonic trap

square well basis states

H. O. basis states

Whats the expectation number of particles in the ground state of the **HARMONIC TRAP**?

$$A_{ij} = \int_{-\infty}^{\infty} v_i^*(\mathbf{r}) u_j(\mathbf{r})\; d^3\mathbf{r}$$

$$\langle \hat{b}_0^\dagger \hat{b}_0\rangle = \langle \sum_i \sum_j A_{0,j}^* A_{0,i} \hat{a}_i^\dagger \hat{a}_j\rangle$$

$$= \sum_i \sum_j A_{0,j}^* A_{0,i} \langle N_0, N_1, 0 \ldots |\hat{a}_i^\dagger \hat{a}_j| N_0, N_1, 0, \ldots\rangle$$

$$= |A_{00}|^2 \langle \hat{a}_0^\dagger \hat{a}_0\rangle + |A_{01}|^2 \langle \hat{a}_1^\dagger \hat{a}_1\rangle$$

$$= |A_{00}|^2 N_0 + |A_{01}|^2 N_1$$

Works for continuous basis too:
Eg: Momentum space operator

$$\hat{\phi}(\mathbf{k}) = \frac{1}{\sqrt{2\pi}} \int \hat{\psi}(\mathbf{r}) e^{-i\mathbf{k}\cdot\mathbf{r}}\; d^3\mathbf{r}$$

Best basis for homogenous systems (ie, no potential)

↕

$$\hat{\psi}(\mathbf{r}) = \frac{1}{\sqrt{2\pi}} \int \hat{\phi}(\mathbf{k}) e^{i\mathbf{k}\cdot\mathbf{r}}\; d^3\mathbf{k}$$

Best basis for strongly interacting systems (because the interparticle interactions are diagonal)

Dynamics: how does the system evolve?
Schrodinger picture:

$$i\hbar\frac{d}{dt}|\Psi\rangle = \hat{\mathcal{H}}|\Psi\rangle$$

$$\hat{\mathcal{H}} = \int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})\hat{H}\hat{\psi}(\mathbf{r})\ d^3\mathbf{r} + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')U(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}')\ d^3\mathbf{r}\,d^3\mathbf{r}'$$

$$|\Psi\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_k=0}^{\infty} C_{n_1,n_2,\ldots,n_k}(t)|n_1,n_2,\ldots,n_k\rangle$$

Equivalent set of equations:

$$|\Psi\rangle = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_k=0}^{\infty} C_{n_1,n_2,\ldots,n_k}(t)|n_1,n_2,\ldots,n_k\rangle$$

$$i\hbar\frac{d}{dt}|\Psi\rangle = i\hbar\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_k=0}^{\infty}\frac{d}{dt}C_{n_1,n_2,\ldots,n_k}(t)|n_1,n_2,\ldots,n_k\rangle$$
$$= \hat{\mathcal{H}}\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_k=0}^{\infty} C_{n_1,n_2,\ldots,n_k}(t)|n_1,n_2,\ldots,n_k\rangle$$

$$i\hbar\frac{d}{dt}C_{m_1,m_2,\ldots,m_k}(t) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty}\cdots\sum_{n_k=0}^{\infty} C_{n_1,n_2,\ldots,n_k}(t)\langle m_1,m_2,\ldots,m_k|\hat{\mathcal{H}}|n_1,n_2,\ldots,n_k\rangle$$

(Equivalent to a giant matrix/vector equation. Can be useful in some simple situations, but is usually too yuk to contemplate, even with a giant computer)

Heisenberg Picture:

$$i\hbar\frac{d}{dt}\hat{\psi}(\mathbf{r}) = \left[\hat{\psi}(\mathbf{r})\,,\ \hat{\mathcal{H}}\right]$$

$$\hat{\mathcal{H}} = \int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})\hat{H}\hat{\psi}(\mathbf{r})\ d^3\mathbf{r} + \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')U(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}')\ d^3\mathbf{r}\,d^3\mathbf{r}'$$

$$i\hbar\frac{d}{dt}\hat{\psi}(\mathbf{r}) = \hat{H}\hat{\psi}(\mathbf{r}) + \left(\int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r}')U(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r}')\ d^3\mathbf{r}'\right)\hat{\psi}(\mathbf{r})$$

The Schrodinger equation is meant to be linear, why is this **non**linear?

Clicker Question:

To calculate the dynamics of some observable, as well as solving the Heisenberg Equation of motion for $\hat{\psi}(\mathbf{r},t)$, we also need...

(a) The initial quantum state $|\Psi(0)\rangle$

(b) The initial condition of $\hat{\psi}(\mathbf{r},0)$

(c) The quantum state for all time $|\Psi(t)\rangle$

(d) All of the above

(e) none of the above

$$i\hbar \frac{d}{dt}\hat{\psi}(\mathbf{r}) = \hat{H}\hat{\psi}(\mathbf{r}) + \left(\int_{-\infty}^{\infty} \hat{\psi}^{\dagger}(\mathbf{r}')U(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r}')\; d^3\mathbf{r}' \right) \hat{\psi}(\mathbf{r})$$

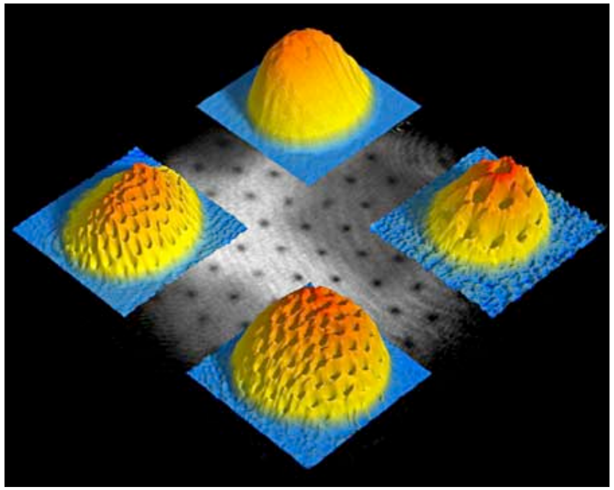
- Tells you everything about the dynamics of the system

- Be careful though: Looks like the schrodinger equation, but it is an OPERATOR equation.

- Still need the state $|\Psi\rangle$ in order to calculate anything.

- Technically, should be equally hard as solving the (many body) schrodinger equation, but there are several useful approximate methods based around this equation.

Example physical system:
Bose-Einstein condensation (BEC) in dilute atomic gas



So what *is* a BEC?

Standard tea-room answer:

– "Macroscopic occupation of a single quantum state"

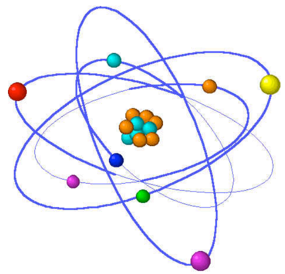
Aren't all collections of atoms in a single quantum state? $|\Psi\rangle$

– Yes, but not necessarily with a large number in some single particle basis mode

$$|\Psi\rangle = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \cdots \sum_{n_k=0}^{\infty} C_{n_1,n_2,\ldots,n_k}(t)|n_1, n_2, \ldots, n_k\rangle$$

– Cooling a collection of particles approaches BEC

It's a bunch of fermions, how come we can treat it as a boson?



Statistical mechanics:

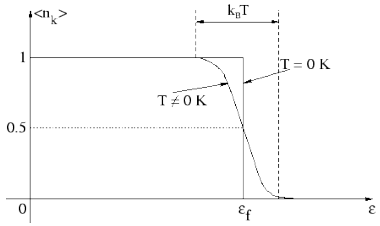
Distinguishable particles: $n(\epsilon) = e^{\frac{-(\epsilon-\mu)}{k_B T}}$ Maxwell-Botsmann distribution

Identical fermions: $n(\epsilon) = \frac{1}{e^{\frac{(\epsilon-\mu)}{k_B T}} + 1}$ Fermi-Dirac distribution

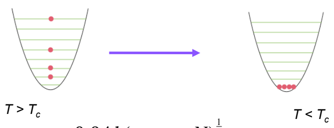
Identical bosons: $n(\epsilon) = \frac{1}{e^{\frac{(\epsilon-\mu)}{k_B T}} - 1}$ Bose-Einstein distribution

Low temperatures:

Fermions:



Bosons:

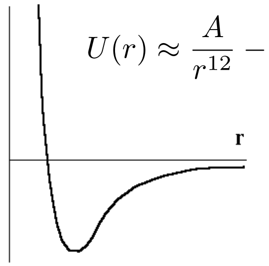


$$T_c \approx \frac{0.94\hbar(\omega_x\omega_y\omega_z N)^{\frac{1}{3}}}{k_B} \approx 300nK$$

Interatomic interaction:

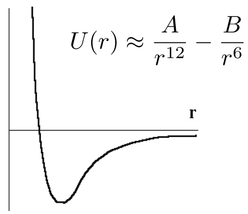
$$\hat{\mathcal{H}} = \int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})\hat{H}\hat{\psi}(\mathbf{r})\ d^3\mathbf{r} + \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')U(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}')\ d^3\mathbf{r}\, d^3\mathbf{r}'$$

What is $U(\mathbf{r}-\mathbf{r}')$



$$U(r) \approx \frac{A}{r^{12}} - \frac{B}{r^6}$$

Can we make this easier to deal with?

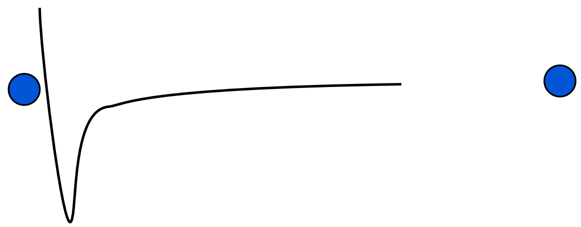


$$U(r) \approx \frac{A}{r^{12}} - \frac{B}{r^6}$$

Short answer:

We can replace it with $U(r-r') = U_0\delta(r)$

More quantitative reasoning:

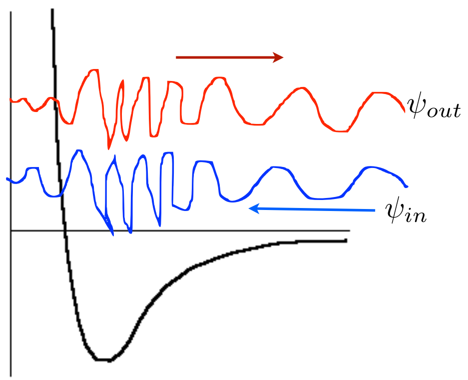


The *effective range* of the potential is short compared to the average inter-particle separation ($\frac{1}{\sqrt[3]{n(r)}}$).

So we are only interested in the effects on the wavefunction at long range.

$$U(r - r') = U_0\delta(r)$$

How do we calculate the constant, Uo?

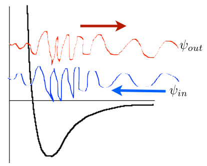


Revision: Scattering theory.

- Calculate the incoming and outgoing wave.

- Use boundary conditions to calculate T and R coeffecients.

What's the reflection coeffecient in this case?



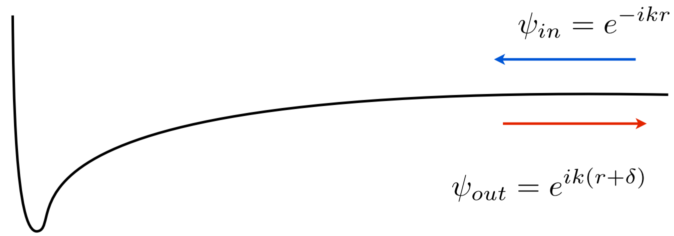
(a) 1

(b) 0

(c) 0.5

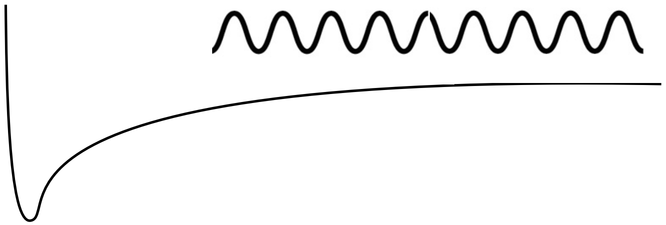
(d) not enough info

Lets look at the incident and reflected waves at large r

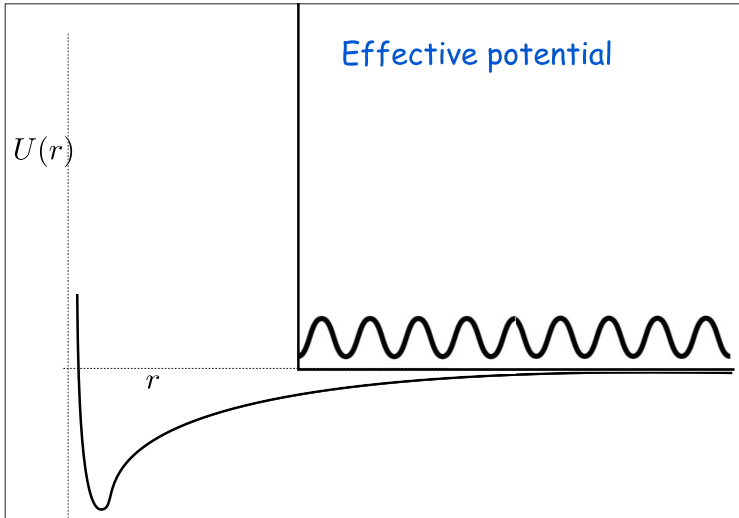


What effect does this phase shift have?

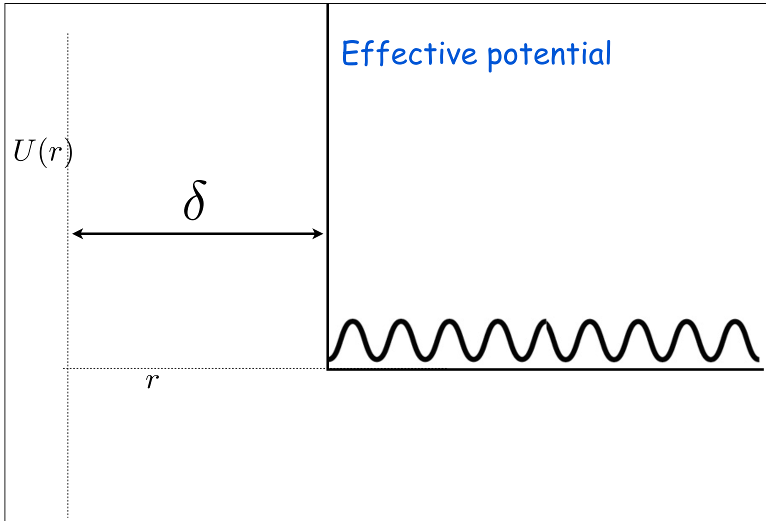
$$\psi = \psi_{in} + \psi_{out}$$
$$= e^{ikr} + e^{-ik(r+\delta)}$$
$$= \cos(kr) + i\sin(kr) + \cos(k(r+\delta)) - i\sin(k(r+\delta))$$



Effective potential



Effective potential



Slightly more precisely: 3D

$$\psi_{in} = e^{ikz} \qquad \psi_{out} = R(r)Y_l^m(\theta,\phi) \approx \frac{e^{ikr}}{r}$$

Calculate the phase shift between ψ_{in} and ψ_{out} in the limit $k \to 0$

Replace $U(r)$ with a *hard sphere* potential of cross-sectional area $\sigma = \frac{4\pi}{k^2}\delta^2 = 4\pi a^2$

Even easier:

Replace $U(r)$ with $U(r) = \frac{4\pi\hbar^2 a}{m}\delta(r)$

a is called the *scattering length*, and is a parameter which is easy to pull out of atomic scattering experiments.

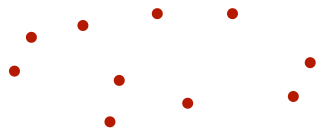
Some length scales:

Atomic size: $\approx a_B \approx 10^{-10}$ m

Scattering length: $\approx 100 a_B \approx 10^{-8}$ m

Mean atomic separation: $\approx \frac{1}{\sqrt[3]{n}} \approx 10^{-7}$ m

De-Broglie wavelength: $\approx$ size of the condensate ≈ 1 mm



Back to BEC:

$$\hat{\mathcal{H}} = \int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r}) \hat{H} \hat{\psi}(\mathbf{r})\, d^3\mathbf{r} + \frac{1}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')U(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}')\, d^3\mathbf{r}\, d^3\mathbf{r}'$$

$$= \int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r}) \hat{H} \hat{\psi}(\mathbf{r})\, d^3\mathbf{r} + \frac{U_0}{2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r}')\delta(\mathbf{r}-\mathbf{r}')\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r}')\, d^3\mathbf{r}\, d^3\mathbf{r}'$$

$$= \int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r}) \hat{H} \hat{\psi}(\mathbf{r})\, d^3\mathbf{r} + \frac{U_0}{2}\int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r})\, d^3\mathbf{r}$$

Yay! looks a fair bit nicer

Equation of motion for the field operator:

$$\hat{\mathcal{H}} = \int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r}) \hat{H} \hat{\psi}(\mathbf{r})\, d^3\mathbf{r} + \frac{U_0}{2}\int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r})\, d^3\mathbf{r}$$

$$i\hbar \frac{d}{dt}\hat{\psi}(\mathbf{r}) = \left[\hat{\psi}(\mathbf{r}),\ \hat{\mathcal{H}}\right]$$

$$\downarrow$$

$$i\hbar \frac{d}{dt}\hat{\psi}(\mathbf{r}) = \hat{H}\hat{\psi}(\mathbf{r}) + U_0 \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r})$$

There are a few things you can calculate with this without too much trouble, but I'll leave that for other people to talk about.

Major classes of Approximations:

Hard to solve, so lets look for some approximations:

Quantum field theory calculations are hard

Approximations ignore either:

– Complexity in the quantum state of each mode

– The number of modes

– Systems with strong interactions

The 'mean field' approximation

- Say we don't care about the correlations, we just care about things like the mean density.

- then all we really are about is $\langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\rangle$

- **What we are throwing away:** the details about the number statistics in each mode.

- **What we are keeping:** The **mean** of the number in each mode, and enough phase to keep track of the dynamics

- Pretty useful, describes most BEC experiments fairly well.

The 'mean field' approximation

Assume the quantum state $|\Psi\rangle$ is such that $\langle \hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\rangle \approx \langle \hat{\psi}^\dagger(\mathbf{r})\rangle\langle\hat{\psi}(\mathbf{r})\rangle$

Then we can calculate everything we want with $\langle\hat{\psi}(\mathbf{r})\rangle$

Evolution:
$$i\hbar\frac{d}{dt}\langle\hat{\psi}(\mathbf{r})\rangle = \hat{H}\langle\hat{\psi}(\mathbf{r})\rangle + U_0\langle\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r})\rangle$$
$$\approx \hat{H}\langle\hat{\psi}(\mathbf{r})\rangle + U_0\langle\hat{\psi}^\dagger(\mathbf{r})\rangle\langle\hat{\psi}(\mathbf{r})\rangle\langle\hat{\psi}(\mathbf{r})\rangle$$

Cleaning up the notation a little by calling $\psi(\mathbf{r}) \equiv \langle\hat{\psi}(\mathbf{r})\rangle$, we arive at the Gross-Pitaevskii Equation (GPE):

$$i\hbar\frac{d}{dt}\psi(\mathbf{r}) = \frac{-\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) + U|\psi(\mathbf{r})|^2\psi(\mathbf{r})$$

The GPE

$$i\hbar\frac{d}{dt}\psi(\mathbf{r}) = \frac{-\hbar^2}{2m}\nabla^2\psi(\mathbf{r}) + V(\mathbf{r})\psi(\mathbf{r}) + U|\psi(\mathbf{r})|^2\psi(\mathbf{r})$$

Easy to solve because ψ is just a single complex function

Looks a lot like the single particle Schrodinger equation, except for the nonlinear bit.

BE CAREFUL: ψ looks like a wavefunction, but it is not a *quantum state* in the true sense. The evolution looks familiar, but the rules of QM such as borns statistical interpretation certainly don't apply.

The nonlinear bit makes the evolution a bit trickier than you are used to. eg: principle of superposition no longer holds.

Clicker Question:

$|\Psi\rangle = |N, 0, \ldots\rangle$. What is $\langle\hat{\psi}\rangle$?

(1) 0

(2) N

(3) $\sqrt{N}u_0(\mathbf{r})$, where u_0 is the appropriate mode function

(4) can't say

So how do we justify this seemingly meaningless approximation?

Answer: assume a different state

$$|\Psi\rangle = e^{-|\alpha|^2} \sum_{n_0=0}^{\infty} \frac{\alpha^{n_0}}{\sqrt{n!}} |n_0, 0, 0, \ldots\rangle \equiv |\alpha\rangle \otimes |0, 0, \ldots\rangle$$

It's not too hard to show $\hat{a}_0|\alpha\rangle = \alpha|\alpha\rangle$

Expand your field operator as $\hat{\psi}(\mathbf{r}, t) = \sum_j \hat{a}_j u_j(\mathbf{r}, t)$

Then $\psi(\mathbf{r}, t) = \langle\hat{\psi}(\mathbf{r}, t)\rangle = \langle\Psi|\sum_j \hat{a}_j u_j(\mathbf{r})|\Psi\rangle = \alpha u_0(\mathbf{r}, t)$

Would also work if we chose a more complicated state:
$|\Psi\rangle = |\alpha_0\rangle \otimes |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \ldots |\alpha_k\rangle$

in that case, $\langle\hat{\psi}\rangle = \sum_j \alpha_j u_j(r, t)$

it's easy to show that this object obeys the same equation of motion.

Problems with this approximation:

$$|\Psi\rangle = |\alpha_0\rangle \otimes |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \ldots |\alpha_k\rangle$$

$$i\hbar\frac{d}{dt}\langle\hat{\psi}(\mathbf{r})\rangle = \hat{H}\langle\hat{\psi}(\mathbf{r})\rangle + U_0\langle\hat{\psi}^\dagger(\mathbf{r})\hat{\psi}(\mathbf{r})\hat{\psi}(\mathbf{r})\rangle$$

$$\approx \hat{H}\langle\hat{\psi}(\mathbf{r})\rangle + U_0\langle\hat{\psi}^\dagger(\mathbf{r})\rangle\langle\hat{\psi}(\mathbf{r})\rangle\langle\hat{\psi}(\mathbf{r})\rangle$$

This state means that this step is completley valid at t=0, but the nonlinear bit ensures that at later times, our state WONT look like this anymore, so it becomes an approximation again.

We have explicitly assumed that there is an uncertain number of particles, yet the mean field wavefunction says there is zero uncertainty in the number of particles.

Lets look at some useful things it can tell us:

What can we do with the GPE?

$$i\hbar\frac{\partial\psi(\mathbf{x})}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V(\mathbf{x}) + U|\psi(\mathbf{x})|^2\right)\psi(\mathbf{x})$$

Can't be right all the time, but it's surprisingly useful

- Spatial behaviour of BEC undergoing only linear processes
  - Evolution in any external potential (including time dependent)
  - Coupling between different internal states
  - Can be used to describe weak BEC excitations, BEC manipulation with optical or magnetic potentials, coupling between internal states, atom lasers, vortices, solitons, wave-guiding, feedback, …

Quantisation of the EM field:

Motivation: There are situations in quantum-atom optics where the quantum state of the EM field matter's when interacting with atoms

Quantisation of the electromagnetic field:

Quick and dirty version

$$\nabla \cdot \mathbf{E} = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

$$\longrightarrow \qquad \mathcal{H} = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)\, d^3\mathbf{r}$$

Simple solution to maxwells equations:

$$\mathbf{E}(z,t) = E_x(z,t)\hat{\mathbf{x}} = q(t)\sin(kz)\hat{\mathbf{x}}$$

$$\nabla \times \mathbf{B} = \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \longrightarrow B_y(z,t) = \frac{1}{kc^2}\dot{q}(t)\cos(kz)$$

$$\mathcal{H} = \frac{1}{2}\int \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)\, d^3\mathbf{r}$$

$$\downarrow$$

$$\mathcal{H} = \frac{1}{2}\left(C_1 q^2 + C_2 \dot{q}^2\right) = \frac{1}{2}\left(C_1 q^2 + C_2 p^2\right)$$

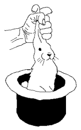
Going from a classical field to a quantum field

Classical:

q and p (and hence E and B) are some numbers that describe the amplitude of the electric and magnetic field.

Quantum:

$\hat{q}$ and $\hat{p}$ (and hence $\hat{E}$ and $\hat{B}$) are some *operators* that operate on some quantum state $|\Psi\rangle$ to describe the amplitude of the electric and magnetic field.

Classical: $[q,\, p] = 0$ ⟶  ⟶ Quantum: $[\hat{q},\, \hat{p}] = i\hbar$

Change variables (again): $\hat{a} \propto \hat{q} + i\hat{p}$ $\qquad [\hat{a}\,,\, \hat{a}^\dagger] = 1$

$$\mathcal{H} = \tfrac{1}{2}\int \left(\epsilon_0 E^2 + \tfrac{1}{\mu_0}B^2\right) d^3\mathbf{r} \longrightarrow \hat{\mathcal{H}} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right)$$

That was assuming just a simple single-mode plane wave. Add in all the modes:

$$\hat{\mathcal{H}} = \sum_j \hbar\omega_j\left(\hat{a}_j^\dagger\hat{a}_j + \frac{1}{2}\right)$$

Electric field operator

$$\hat{\mathbf{E}}(\mathbf{r},t) = \sum_{\mathbf{k}} \hat{\epsilon}_{\mathbf{k}}\mathcal{E}_{\mathbf{k}}\left(\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

$$\hat{\mathbf{B}}(\mathbf{r},t) = -i\epsilon_0 c\sum_{\mathbf{k}} \mathbf{k}\times\hat{\epsilon}_{\mathbf{k}}\mathcal{E}_{\mathbf{k}}\left(\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

$$\mathcal{E}_k = \sqrt{\frac{\hbar\omega_k}{\epsilon_0 V}}$$

$$\hat{\mathcal{H}} = \tfrac{1}{2}\int \left(\epsilon_0 \hat{E}^2 + \tfrac{1}{\mu_0}\hat{B}^2\right) d^3\mathbf{r} \quad = \sum_j \hbar\omega_j\left(\hat{a}_j^\dagger\hat{a}_j + \frac{1}{2}\right)$$

Clicker Question:

$$\hat{\mathbf{E}}(\mathbf{r},t) = \sum_{\mathbf{k}} \hat{\epsilon}_{\mathbf{k}}\mathcal{E}_{\mathbf{k}}\left(\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}\right) \qquad \hat{\mathbf{B}}(\mathbf{r},t) = -i\epsilon_0 c\sum_{\mathbf{k}} \mathbf{k}\times\hat{\epsilon}_{\mathbf{k}}\mathcal{E}_{\mathbf{k}}\left(\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

Are E and B observable quantities?

(1) Yes, they both are

(2) No, neither of them is

(3) E is but B isn't

(4) B is but E isn't

(5) not enough info

Clicker Question:

$$\hat{\mathbf{E}}(\mathbf{r},t) = \sum_{\mathbf{k}} \hat{\epsilon}_{\mathbf{k}}\mathcal{E}_{\mathbf{k}}\left(\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}\right) \qquad \hat{\mathbf{B}}(\mathbf{r},t) = -i\epsilon_0 c\sum_{\mathbf{k}} \mathbf{k}\times\hat{\epsilon}_{\mathbf{k}}\mathcal{E}_{\mathbf{k}}\left(\hat{a}_{\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{r}} - \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}}\right)$$

Are E and B *Compatible Observables?*

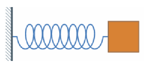
(1) Yes

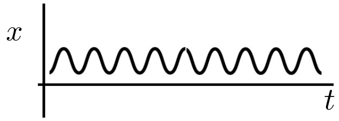
(2) No

(3) Not enough info

Mass on a spring:

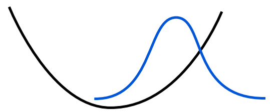
Classical:

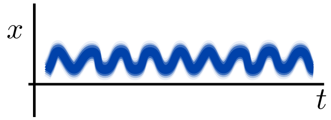




$$x(t) \propto a(t) + a^*(t)$$

Quantum:

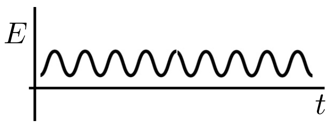




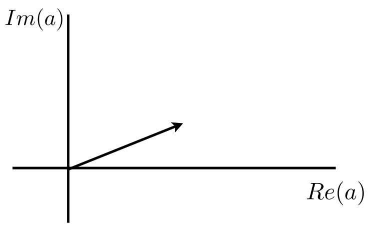
$$\hat{x}(t) \propto \hat{a}(t) + \hat{a}^\dagger(t)$$

Electric field

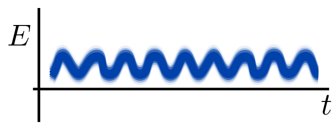
Classical:



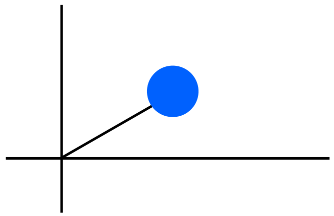
$$E(t) \propto a(t) + a^*(t)$$



Quantum:



$$\hat{E}(t) \propto \hat{a}(t) + \hat{a}^\dagger(t)$$



$$\hat{\mathbf{E}}(\mathbf{r},t) = \sum_{\mathbf{k}} \hat{\epsilon}_{\mathbf{k}} \mathcal{E}_{\mathbf{k}} \left(\hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{a}^\dagger_{\mathbf{k}} e^{-i\mathbf{k}\cdot\mathbf{r}} \right)$$

Exercise:

Calculate the electric field for a single mode with 100 photons in it

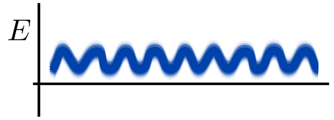
Moral: a Fock state isn't a good approximation to a classical EM wave with well defined amplitude and phase.

So what state is?

Coherent state

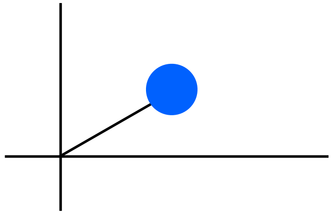
$$|\alpha\rangle = e^{\frac{-|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Quantum:



It's the state generated when you couple a classical current to the EM vacuum

$$\hat{E}(t) \propto \hat{a}(t) + \hat{a}^\dagger(t)$$



eigenstate of the annihilation operator:

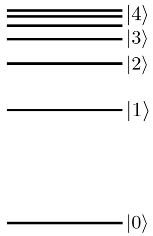
$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

Atoms interacting with light

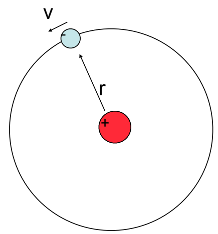
$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{atoms} + \hat{\mathcal{H}}_{light} + \hat{\mathcal{H}}_{atom-light}$$

$$\hat{\mathcal{H}}_{atom-light} = \hat{\mathbf{d}} \cdot \hat{\mathbf{E}}$$

Revision: Dipole moment of a single 2-level atom:



$$\hat{\mathbf{d}} = e\mathbf{r}$$



$$|\Psi\rangle = \sum_n c_n|n\rangle \approx c_0|0\rangle + c_1|1\rangle$$

$$\hat{\mathbf{d}} = e\mathbf{r} \;\; = \hat{1}e\mathbf{r}\hat{1}$$

$$= (|0\rangle\langle 0| + |1\rangle\langle 1|)\, e\mathbf{r}\, (|0\rangle\langle 0| + |1\rangle\langle 1|)$$

$$= \mathbf{d}_{10}\, |1\rangle\langle 0| + \mathbf{d}_{01}\, |0\rangle\langle 1| \qquad \mathbf{d}_{ij} = \langle i|e\mathbf{r}|j\rangle$$

$$= \mathbf{d}\,(\sigma_+ + \sigma_-)$$

ok cool. Now what about many atoms?

$$\hat{\mathbf{d}}_{manybody} = \int_{-\infty}^{\infty} \hat{\psi}^\dagger(\mathbf{r})\hat{\mathbf{d}}\hat{\psi}(\mathbf{r})\; d^3\mathbf{r}$$

What does a general (many body) state look like?

E_1 ——— $|1\rangle$

——— $|0\rangle$

$$|\Psi\rangle = \sum_{n_{a_1}=0}^{\infty} \cdots \sum_{n_{a_k}=0}^{\infty} \sum_{n_{b_1}=0}^{\infty} \cdots \sum_{n_{b_2}=0}^{\infty} C_{n_{a_1}\ldots n_{a_k} n_{b_1} \ldots n_{b_k}} |n_{a_1}, \ldots, n_{a_k}, n_{b_1}, \ldots n_{b_k}\rangle$$

Many-body dipole operator

E_1 ——— $|1\rangle$

——— $|0\rangle$

$\hat{\psi}_a$ annihilates a state $|0\rangle$ atom

$\hat{\psi}_b$ annihilates a state $|1\rangle$ atom

$$\hat{\mathbf{d}}_{manybody} = \int_{-\infty}^{\infty} \hat{\psi}_a^\dagger(\mathbf{r}) \hat{\mathbf{d}} \hat{\psi}_a(\mathbf{r}) \; d^3\mathbf{r} + \int_{-\infty}^{\infty} \hat{\psi}_b^\dagger(\mathbf{r}) \hat{\mathbf{d}} \hat{\psi}_b(\mathbf{r}) \; d^3\mathbf{r}$$

$$\hat{\mathbf{d}} = \mathbf{d}\,(\sigma_+ + \sigma_-)$$

$$\hat{\mathbf{d}}_{manybody} = \mathbf{d} \int_{-\infty}^{\infty} \left(\hat{\psi}_b^\dagger \hat{\psi}_a + \hat{\psi}_a^\dagger \hat{\psi}_b \right) \; d^3\mathbf{r}$$

Atom-Field interaction

$$\hat{\mathcal{H}} = \hat{\mathcal{H}}_{atoms} + \hat{\mathcal{H}}_{light} + \hat{\mathbf{d}} \cdot \hat{\mathbf{E}}$$

$$\hat{\mathbf{E}}(\mathbf{r}, t) = \sum_{\mathbf{k}} \hat{\epsilon}_{\mathbf{k}} \mathcal{E}_{\mathbf{k}} \left(\hat{a}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{a}_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}} \right) \qquad \hat{\mathbf{d}} = \mathbf{d} \int_{-\infty}^{\infty} \left(\hat{\psi}_b^\dagger \hat{\psi}_a + \hat{\psi}_a^\dagger \hat{\psi}_b \right) \; d^3\mathbf{r}$$

$$\hat{\mathcal{H}}_{atom-light} = \sum_{\mathbf{k}} g_{\mathbf{k}} \int_{-\infty}^{\infty} \left(\hat{a}_{\mathbf{k}} \hat{\psi}_b^\dagger \hat{\psi}_a e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{a}_{\mathbf{k}} \hat{\psi}_b \hat{\psi}_a^\dagger e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{a}_{\mathbf{k}}^\dagger \hat{\psi}_b^\dagger \hat{\psi}_a e^{-i\mathbf{k}\cdot\mathbf{r}} + \hat{a}_{\mathbf{k}}^\dagger \hat{\psi}_b \hat{\psi}_a^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}} \right) \; d^3\mathbf{r}$$

↓ rotating wave approximation

$$\hat{\mathcal{H}}_{atom-light} = \sum_{\mathbf{k}} g_{\mathbf{k}} \int_{-\infty}^{\infty} \left(\hat{a}_{\mathbf{k}} \hat{\psi}_b^\dagger \hat{\psi}_a e^{i\mathbf{k}\cdot\mathbf{r}} + \hat{a}_{\mathbf{k}}^\dagger \hat{\psi}_b \hat{\psi}_a^\dagger e^{-i\mathbf{k}\cdot\mathbf{r}} \right) \; d^3\mathbf{r}$$

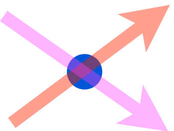
What about in momentum space?

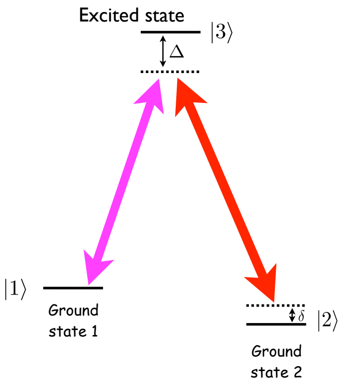
$$\hat{\mathcal{H}}_{atom-light} = g \int_{-\infty}^{\infty} \left(\hat{a}_{\mathbf{k}_0} \hat{\psi}_b^\dagger(\mathbf{r}) \hat{\psi}_a(\mathbf{r}) e^{i\mathbf{k}_0\cdot\mathbf{r}} + \hat{a}_{\mathbf{k}_0}^\dagger \hat{\psi}_b(\mathbf{r}) \hat{\psi}_a^\dagger(\mathbf{r}) e^{-i\mathbf{k}_0\cdot\mathbf{r}} \right) \; d^3\mathbf{r}$$

↓

$$\hat{\mathcal{H}}_{atom-light} = g \int_{-\infty}^{\infty} \left(\hat{a}_{\mathbf{k}_0} \hat{\phi}_b^\dagger(\mathbf{k} + \mathbf{k}_0) \hat{\phi}_a(\mathbf{k}) + \hat{a}_{\mathbf{k}_0}^\dagger \hat{\phi}_b(\mathbf{k} + \mathbf{k}_0) \hat{\phi}_a^\dagger(\mathbf{k}) \right) \; d\mathbf{k}^3$$

Fun with atoms and light:



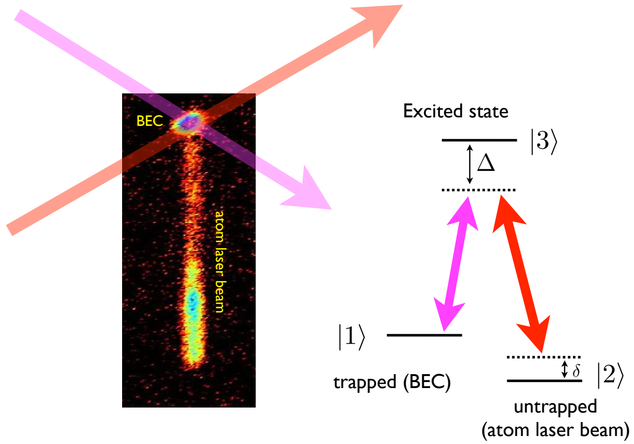


$$\hat{\mathcal{H}}_{atom-light} = g_1 \int_{-\infty}^{\infty} \left(\hat{a}_1 \hat{\phi}_3^\dagger(\mathbf{k}+\mathbf{k}_1)\hat{\phi}_1(\mathbf{k}) + \hat{a}_1^\dagger \hat{\phi}_3(\mathbf{k}+\mathbf{k}_1)\hat{\phi}_1^\dagger(\mathbf{k}) \right) d\mathbf{k}^3$$

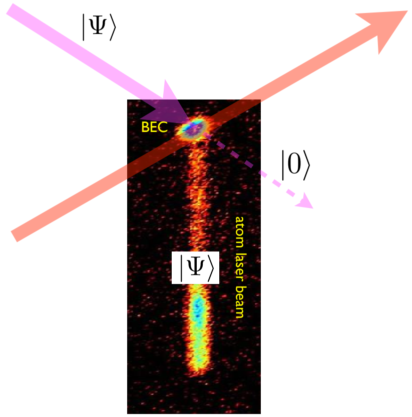
$$+ g_2 \int_{-\infty}^{\infty} \left(\hat{a}_2 \hat{\phi}_3^\dagger(\mathbf{k}+\mathbf{k}_2)\hat{\phi}_2(\mathbf{k}) + \hat{a}_2^\dagger \hat{\phi}_3(\mathbf{k}+\mathbf{k}_2)\hat{\phi}_2^\dagger(\mathbf{k}) \right) d\mathbf{k}^3$$

$$\approx \frac{g_1 g_2}{\Delta} \int_{-\infty}^{\infty} \left(\hat{a}_1 \hat{a}_2^\dagger \hat{\phi}_1(\mathbf{k})\hat{\phi}_2^\dagger(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2) + \hat{a}_2 \hat{a}_1^\dagger \hat{\phi}_2(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\hat{\phi}_1^\dagger(\mathbf{k}) \right) d\mathbf{k}^3$$

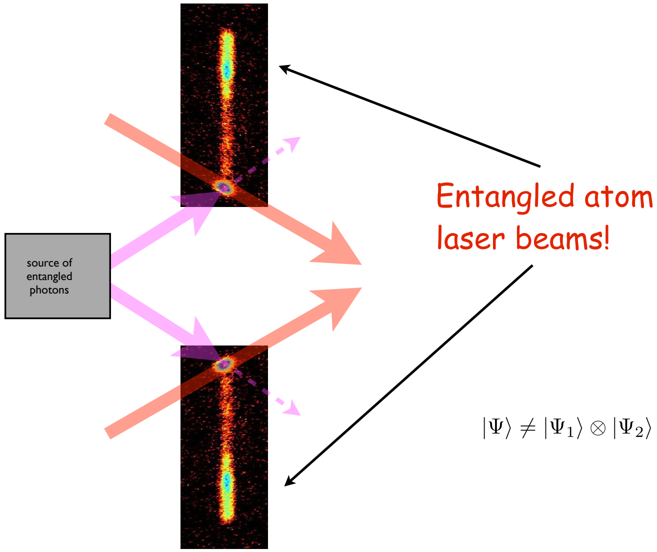
The atom laser



Quantum state transfer

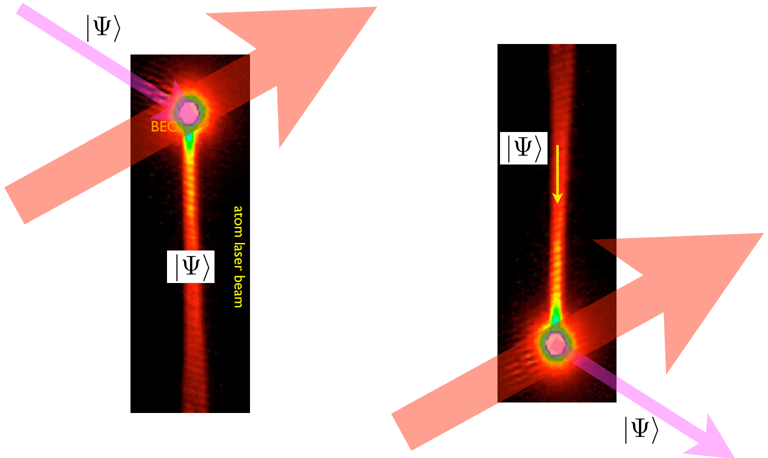


Entangled atom laser beams:

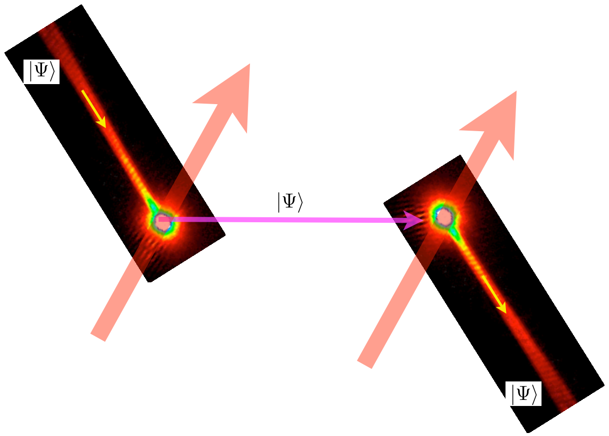


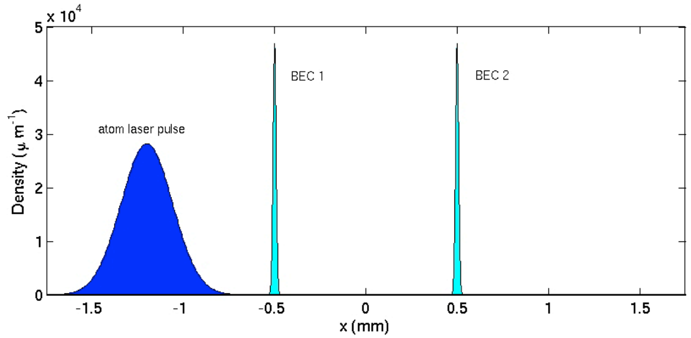
$$|\Psi\rangle \neq |\Psi_1\rangle \otimes |\Psi_2\rangle$$

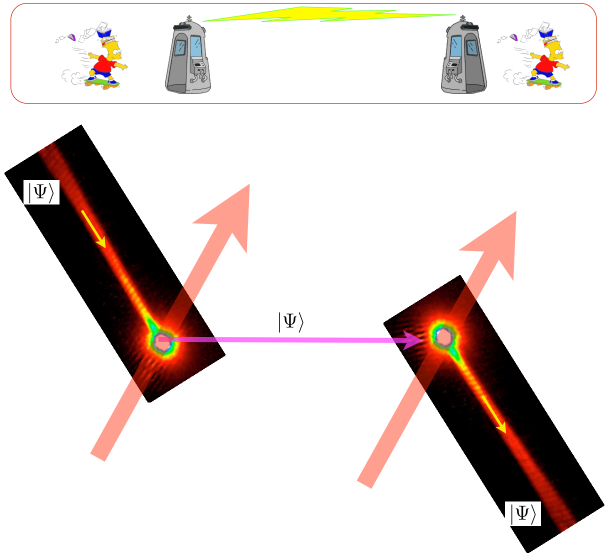
$$\hat{\mathcal{H}} = \frac{g_1 g_2}{\Delta}\int_{-\infty}^{\infty}\left(\hat a_1 \hat a_2^\dagger \hat\phi_1(\mathbf{k})\hat\phi_2^\dagger(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2) + \hat a_2 \hat a_1^\dagger \hat\phi_2(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\hat\phi_1^\dagger(\mathbf{k})\right) d\mathbf{k}^3$$



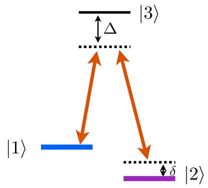
$$\hat{\mathcal{H}} = \frac{g_1 g_2}{\Delta}\int_{-\infty}^{\infty}\left(\hat a_1 \hat a_2^\dagger \hat\phi_1(\mathbf{k})\hat\phi_2^\dagger(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2) + \hat a_2 \hat a_1^\dagger \hat\phi_2(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\hat\phi_1^\dagger(\mathbf{k})\right) d\mathbf{k}^3$$



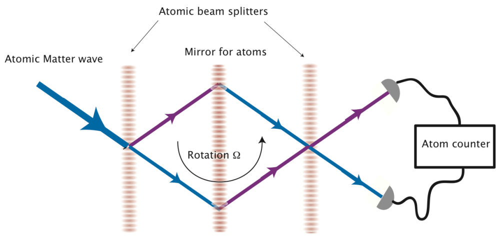


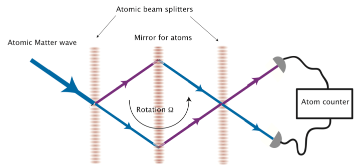


Atom Interferometry:



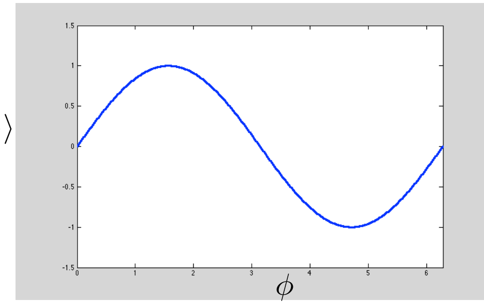
$$\hat{\mathcal{H}} \approx \int_{-\infty}^{\infty} \left(\frac{\Omega_1\Omega_2^*}{\Delta}\hat{\phi}_1(\mathbf{k})\hat{\phi}_2^\dagger(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2) + \frac{\Omega_1^*\Omega_2}{\Delta}\hat{\phi}_2(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2)\hat{\phi}_1^\dagger(\mathbf{k})\right) d\mathbf{k}^3$$



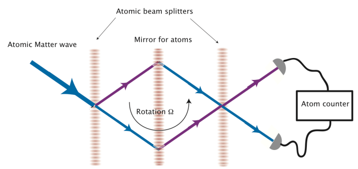


$$\phi_{atoms} = \left(\frac{2m}{\hbar}\right)\mathbf{\Omega}\times\mathbf{A}$$

$\langle(\hat{N}_1 - \hat{N}_2)\rangle$

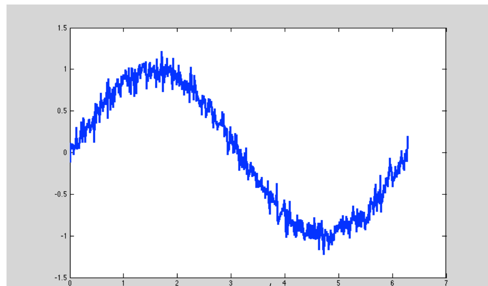


ϕ



$$\phi_{atoms} = \left(\frac{2m}{\hbar}\right)\mathbf{\Omega}\times\mathbf{A}$$

$\hat{N}_1 - \hat{N}_2$



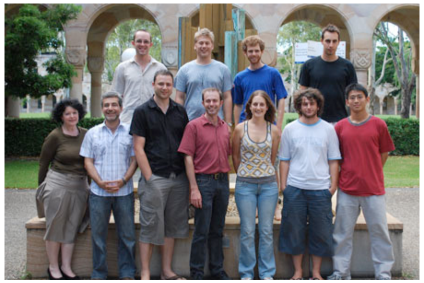
ϕ

$$\Delta\phi = \frac{\sqrt{V(N_1-N_2)}}{\left|\frac{d(N_1-N_2)}{d\phi}\right|} = \frac{1}{\sqrt{N}}$$

Research Projects with me:

- Quantum Enhanced atom interferometry
- Atom-light entanglement
- Tests of quantum gravity and decoherence

Other atom optics at UQ:



Theory:

- Far from equilibrium quantum systems
- Entanglement
- Fermions
- Precision measurement
- Atomic Physics
- Open quantum systems
- Quantum Optics

Experiment:

- BEC in ring-traps
- Atom interferometry
- Quantum Simulators

the end