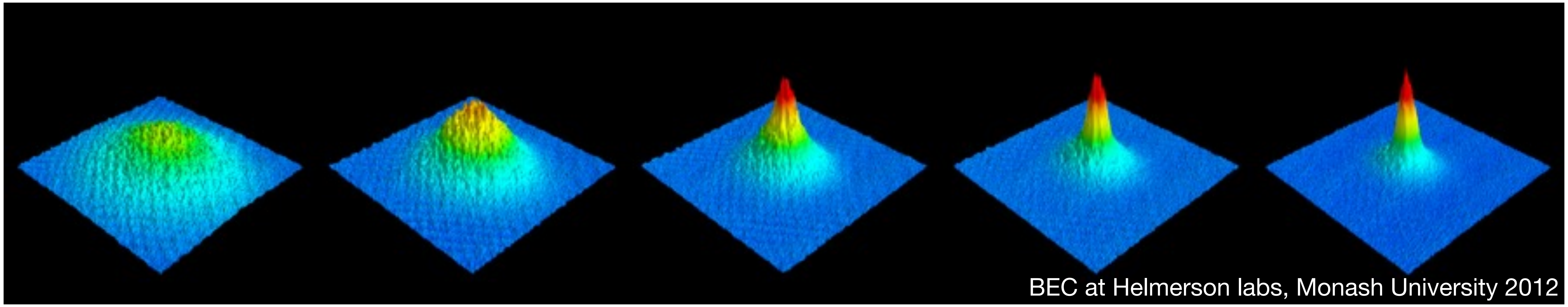
Victorian Summer School in Ultracold Physics

25th June – 6th July 2012
Melbourne, Victoria, Australia

A brief introduction to Bose-Einstein condensation in dilute gases

Tapio Simula



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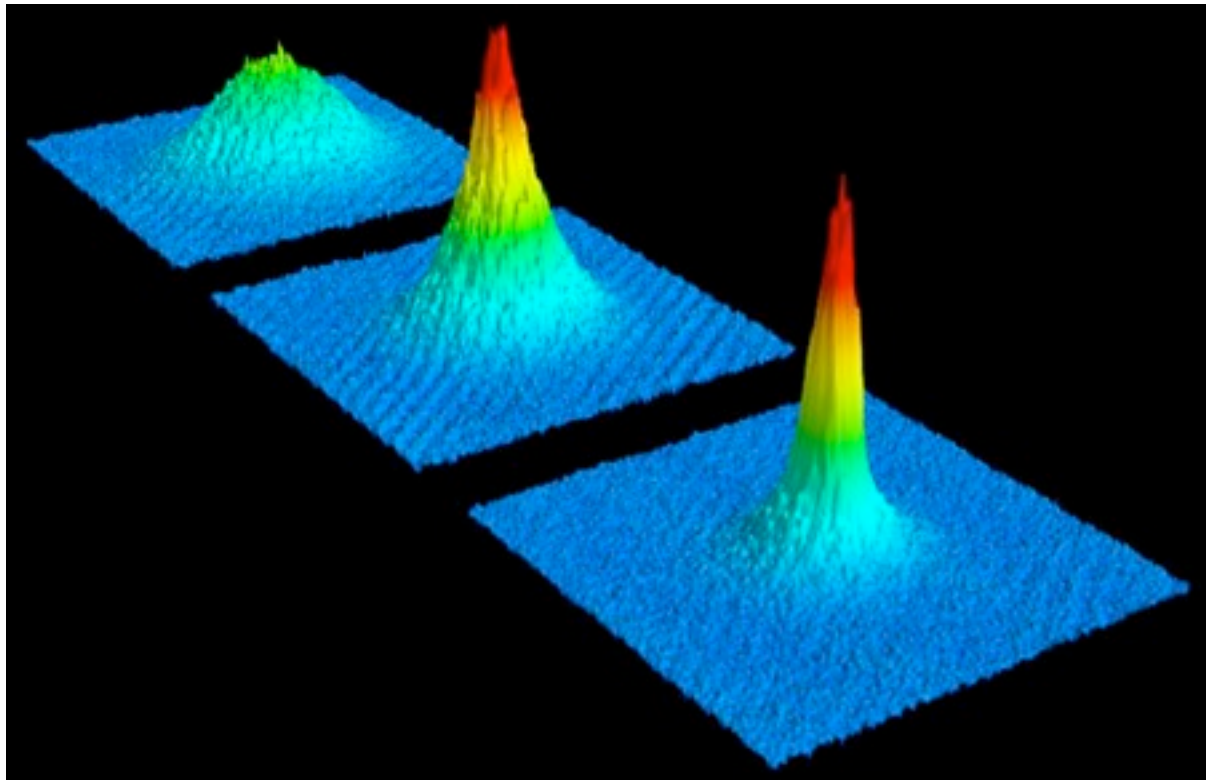
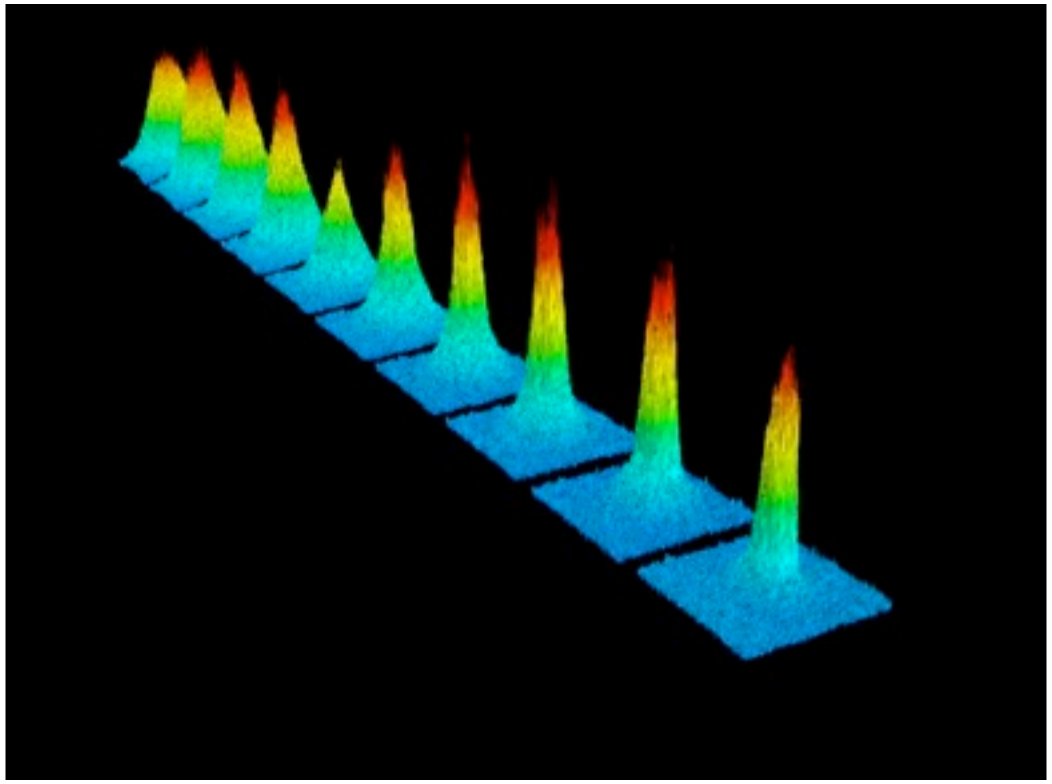
Dr Lincoln Turner :: lincoln.turner@monash.edu experiments

Dr Tapio Simula :: tapio.simula@monash.edu theory



BEC at Helmerson lab



BEC at Turner lab

Lecture 1: **BEC 1**

- *Bose-Einstein condensation*
- *Gross-Pitaevskii equation*
- *Thomas-Fermi approximation*

Lecture 2: **BEC 2**

- *hydrodynamic formulation*
- *collective excitations*
- *superfluidity*

Lecture 3: **Vortices 1**

- *topology of the order parameter*
- *structure and stability of vortices*
- *vortex dynamics*

Lecture 4: **Vortices 2**

- *spin-vortices in spinor BECs*
- *vortices and electromagnetism*
- *quantum turbulence*

Rev. Mod. Phys. 81, 647–691 (2009)
A. L. Fetter
Rotating trapped Bose-Einstein condensates

C.J. Pethick and H. Smith,
“Bose-Einstein Condensation in Dilute Gases”,
(Cambridge University Press 2001).

L. Pitaevskii and S. Stringari,
“Bose-Einstein Condensation”,
(Oxford University Press 2003).

A.J. Leggett,
“Quantum Liquids”,
(Oxford University Press 2006).

A. Griffin, T. Nikuni and E. Zaremba,
“Bose-Condensed Gases at Finite Temperatures”,
(Cambridge University Press 2009).

M. Ueda,
“Fundamentals and new frontiers of Bose-Einstein condensation”,
(World Scientific 2010).

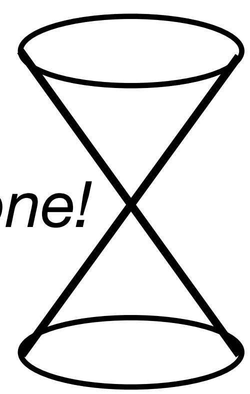
Reviews of Modern Physics:
Nobel lectures: 1997, 2001, 2003

Rev. Mod. Phys. reviews by:
- Dalfovo, Pitaevskii and Stringari
- Leggett
- Fetter
- Dalibard
- ...

\+ other review and research journals

Doctoral Theses!

use the citation light cone!



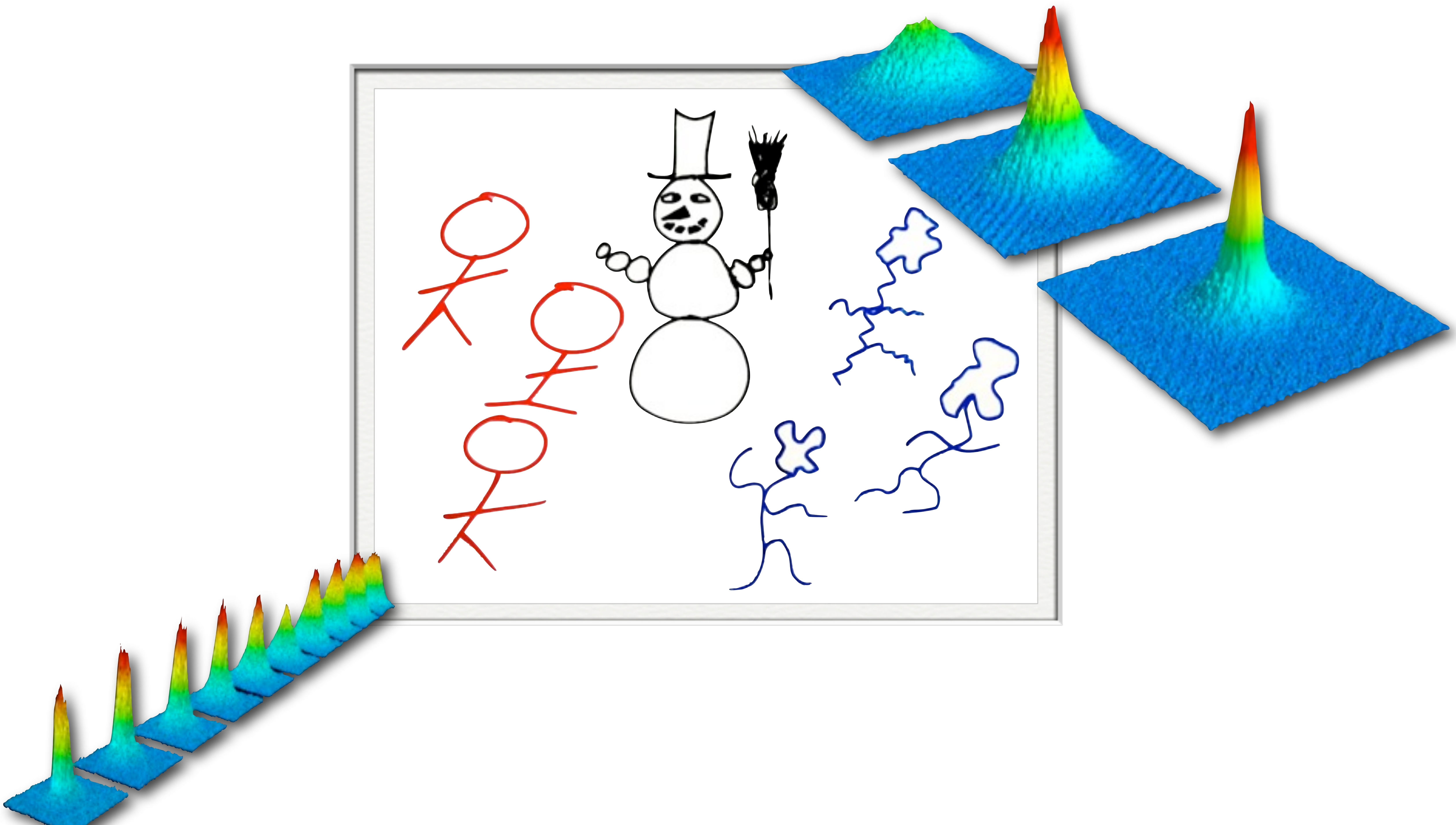
http://www.colorado.edu/physics/2000/bec/

Absolute Zero
(documentary)

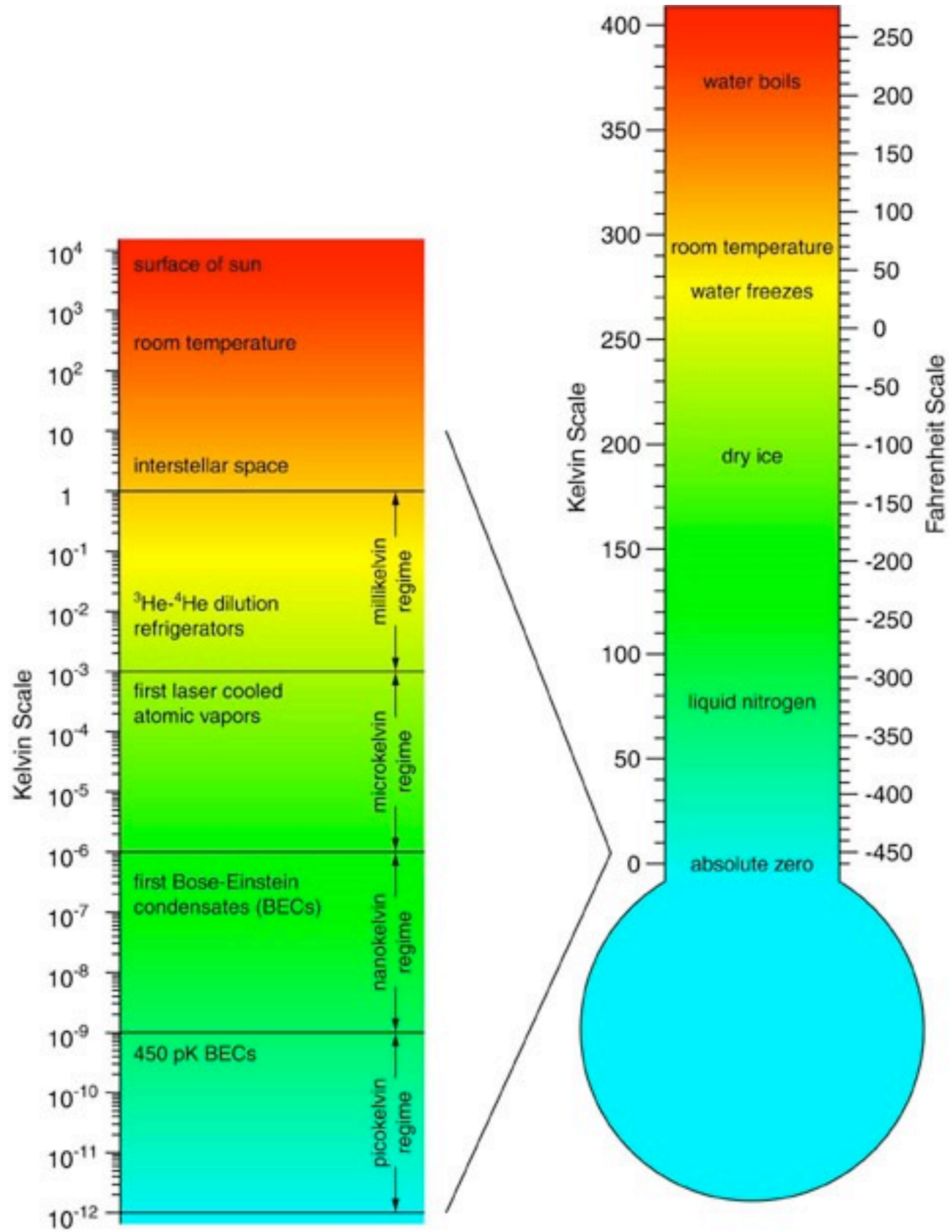
http://youtu.be/y2jSv8PDDwA

Lecture 1

where bosons meet absolute zero and go quantum

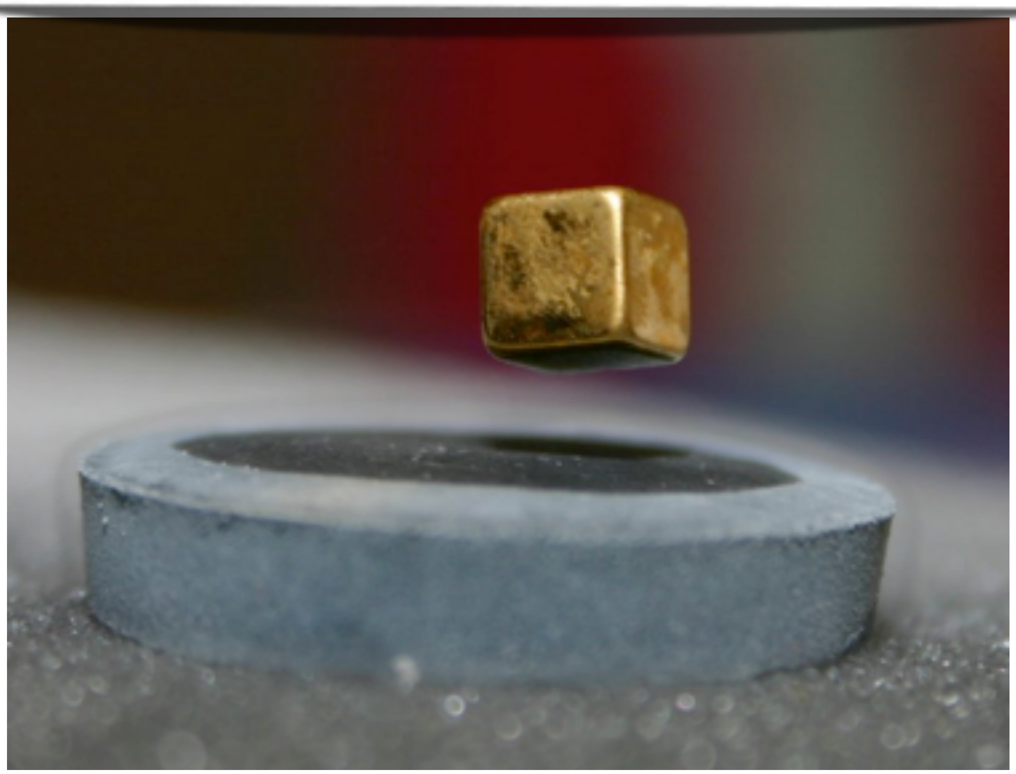


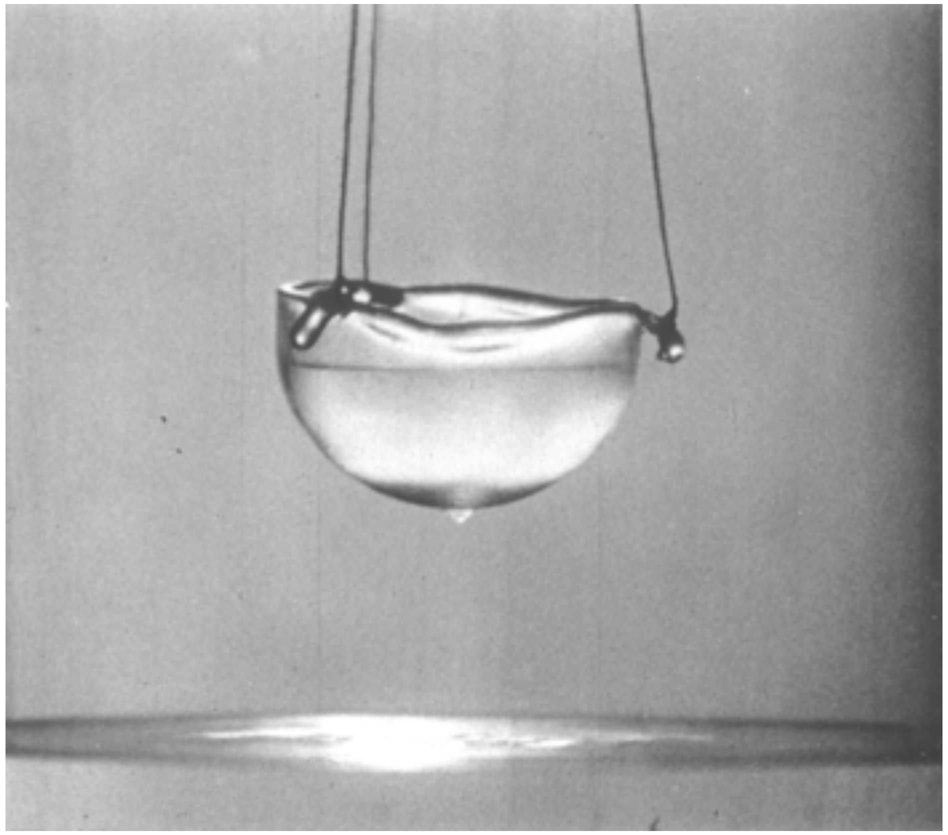
How cold is cold?



A piece of history

liquid helium	1908
supraconductivity	1911
theory papers by B&E	1924
superfluid helium	1937
BEC in magnetic traps	1995





http://en.wikipedia.org/wiki/Superfluid

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Plancks Gesetz und Lichtquantenhypothese.

Von **Bose** (Dacca-University, Indien).

(Eingegangen am 2. Juli 1924.)

Der Phasenraum eines Lichtquants in bezug auf ein gegebenes Volumen wird in „Zellen“ von der Größe h^3 aufgeteilt. Die Zahl der möglichen Verteilungen der Lichtquanten einer makroskopisch definierten Strahlung unter diese Zellen liefert die Entropie und damit alle thermodynamischen Eigenschaften der Strahlung.

derivation of Planck's radiation law

$$E(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{e^{h\nu/kT} - 1}$$

(Übersetzt von A. Einstein.)

Anmerkung des Übersetzers. Boses Ableitung der Planckschen Formel bedeutet nach meiner Meinung einen wichtigen Fortschritt. Die hier benutzte Methode liefert auch die Quantentheorie des idealen Gases, wie ich an anderer Stelle ausführen will.

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SITZUNGSBERICHTE

DER PREUSSISCHEN

AKADEMIE DER WISSENSCHAFTEN.

1924

XXII. Gesamtsitzung. 10. Juli.

Vorsitzender Sekretar: Hr. Lüders.

1. Hr. Penck sprach über das Hauptproblem der physischen Anthropogeographie.

Es besteht in den Beziehungen zwischen Erdoberfläche und Mensch, welche durch dessen Ernährungsbedürfnis hergestellt werden. Zur Befriedigung desselben steht eine Anbaufläche von ziemlich enger Begrenzung zur Verfügung. Unter den besten heutigen Produktionsverhältnissen könnten 8—9 Milliarden Menschen auf der Erde existieren, welche Zahl nach der jetzigen Vermehrungsrate der Menschheit in 300 Jahren erreicht wird.

2. Hr. von Harnack legte eine Abhandlung vor: »Die Reden Pauls von Samosata an Sabinus (Zenobia?) und seine Christologie.«

In seiner soeben (Leipzig 1924) erschienenen großen Monographie über Paul von Samosata erklärt Hr. Loofs, einen echten Kern anerkennend, die überlieferten Fragmente aus den »Reden an Sabinus« für unecht, läßt sie bei seiner Feststellung der Christologie des häretischen Bischofs fast ganz beiseite und stellt es in Abrede, daß diese Christologie eine dynamistische bzw. adoptianische sei. Demgegenüber wird in dieser Abhandlung die Echtheit der Fragmente und die bisher gültige Auffassung der Christologie Pauls verteidigt.

3. Hr. Heider sprach über den Zahnwechsel bei polychäten Anneliden.

Er bezieht sich auf eine Beobachtung von Ehlers über Kieferersatz bei »*Eunice harassii*« und knüpft daran allgemeine Bemerkungen über Zahnersatz, Borstenersatz und Häutungsvorgänge bei Polychäten.

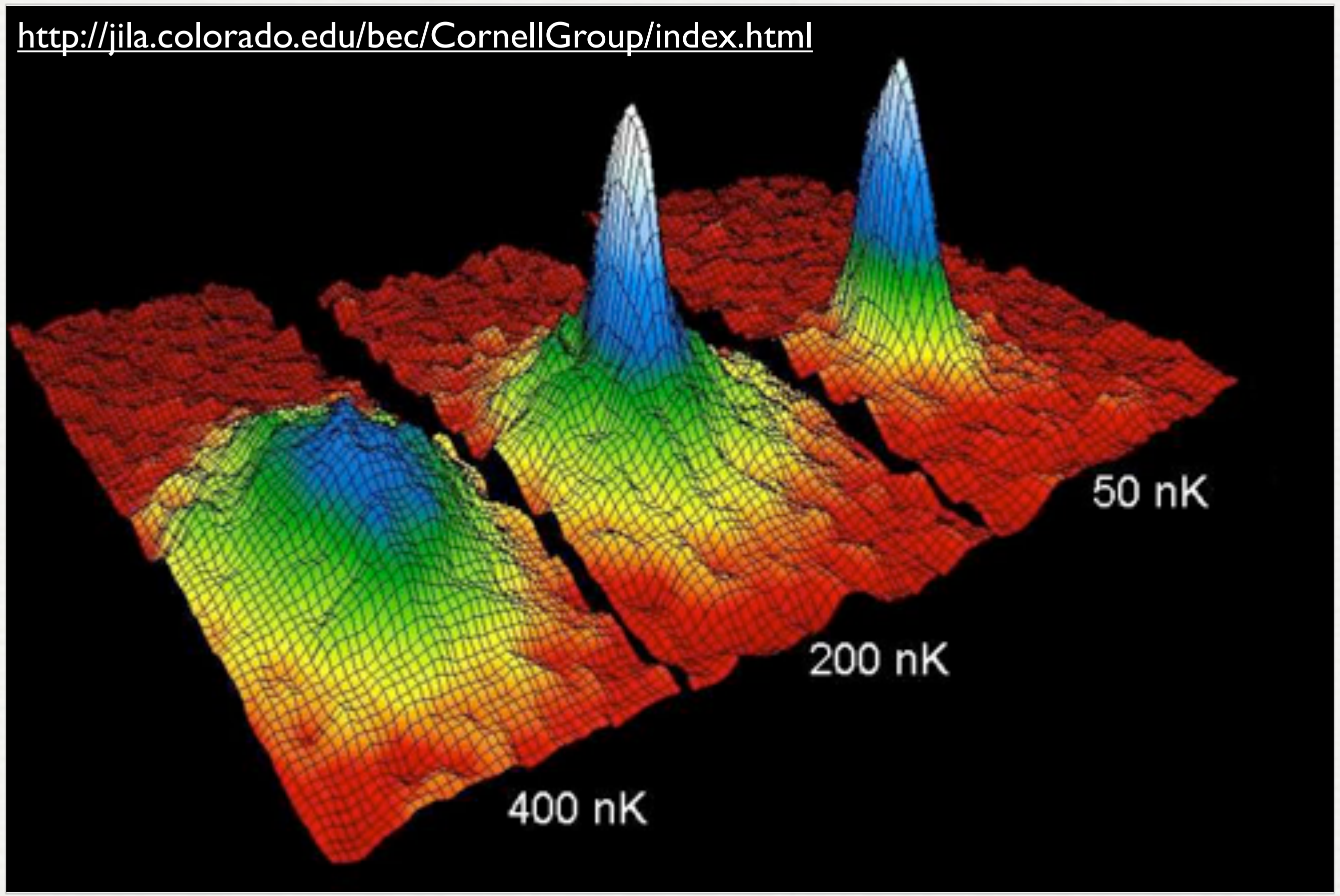
4. Hr. Einstein legte einen Aufsatz vor: »Quantentheorie des einatomigen idealen Gases.«

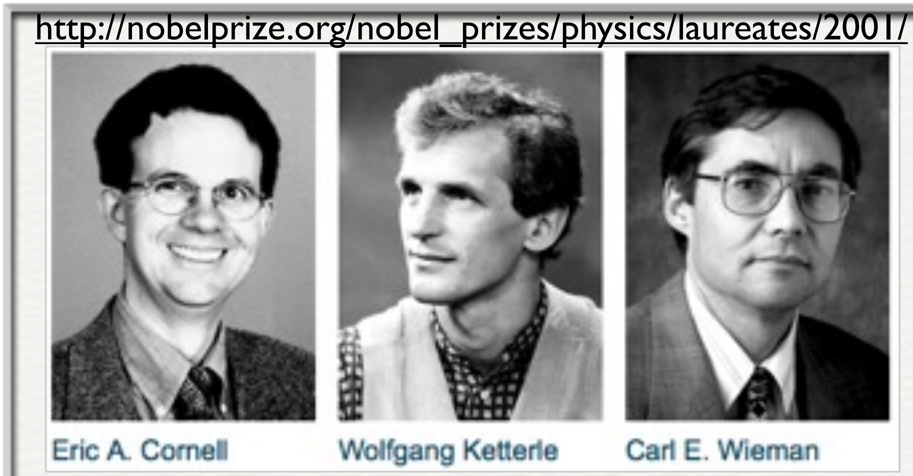
Die Methode, auf Grund welcher Hr. Bose die Plancksche Strahlungsformel abgeleitet hat, läßt sich auch auf ideale Gase anwenden. Man findet so eine Abweichung von der klassischen Zustandsgleichung idealer Gase bei tiefen Temperaturen (Entartung). Am Schlusse wird ein Paradoxon angegeben, das die Gültigkeit der gefundenen Gesetze als zweifelhaft erscheinen läßt.

5. Hr. Norden legte den Bericht der Kommission für den Thesaurus linguae Latinae über die Zeit vom 1. April 1922 bis 31. März 1924 vor.

Bose-Einstein condensate was born at JILA in 1995

(of the type envisaged by A.E. in 1924)





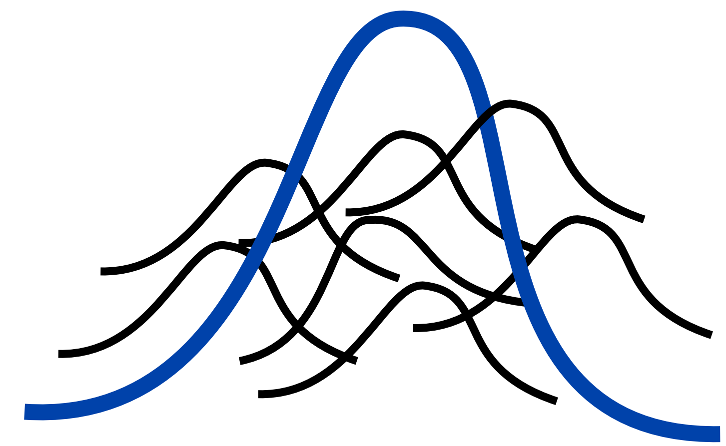
FEATURING:
- confinement + isolation from environment
- ultra-cold temperatures (to unveil QM)
- super-low density (to prevent solidification)
- mega-high phase space density
  (to enable wavefunction overlap)

BEC in other systems:
- superconductors (solid)
- superfluid helium (liquid)
  - ^{4}He boson
  - ^{3}He fermion
- degenerate Fermi gas (gas)
- exciton-polaritons, magnons
- photons as of 2010 !!!
- more to come...

in fermionic systems Cooper pairing yields bosons which can undergo a BEC

Expect BEC when phase space becomes dense:

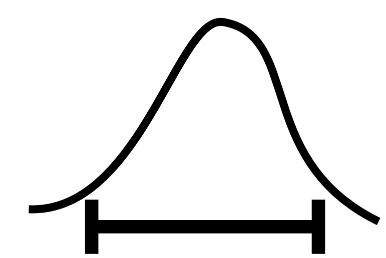
$$\boxed{n\lambda^3 \gtrsim 1}$$

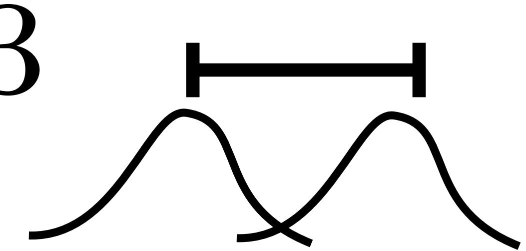


wave size of a particle

$$\frac{N}{V}\left(\frac{h}{\sqrt{2\pi m k_B T}}\right)^3$$

$$\frac{\lambda^3}{d^3}$$





interparticle separation

At what temperature might liquid helium form a Bose-Einstein condensate?

$$\frac{N}{V}\left(\frac{h}{\sqrt{2\pi m k_B T}}\right)^3$$

$$\mathrm{amu} = 1.6 \times 10^{-27}\mathrm{kg}$$

$$h = 6.6 \times 10^{-34}\mathrm{m}^2\ \mathrm{kg/s}$$

$$k_\mathrm{B} = 1.4 \times 10^{-23}\mathrm{J/K}$$

Could Jupiter be a BEC?

Why BEC?

two complementary frameworks to describe corpuscles: particles and waves

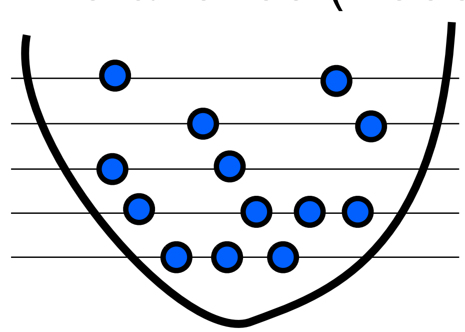
$$\lambda = \frac{h}{p}$$

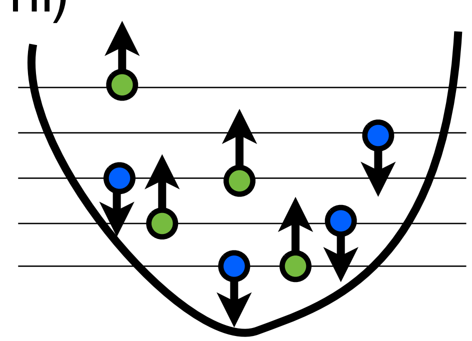
$$\frac{p^2}{2m} = k_{\mathrm{B}}T$$

1. quantum mechanics becomes important when

$$\lambda \gtrsim d \longrightarrow k_{\mathrm{B}}T \lesssim n^{2/3}\hbar^2/m$$

2. enter quantum statistics (Bose or Fermi)





$$n\lambda^3 \gtrsim 1$$

wavefunctions begin to overlap

3. in equilibrium formation of a zero-entropy BEC state minimizes free energy!

BEC definition

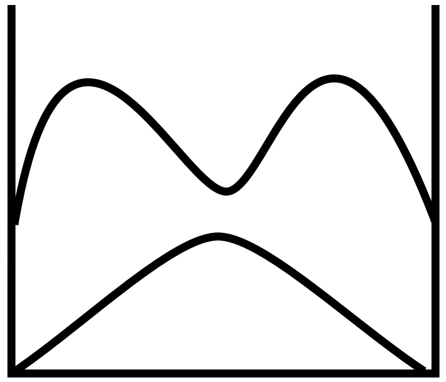
“a macroscopic number of particles occupies a single one-particle state”

BEC characteristic

off-diagonal-long-range-order (ODLRO)

$$\lim_{|r-r'|\to\infty} \langle \Psi^\dagger(rt)\Psi(r't)\rangle \to const.$$

case study :: an ideal Bose gas in a 3D box



Hamiltonian contains only Laplacian and hence the Schrödinger equation is

$$-\frac{\hbar^2}{2m}\nabla^2\varphi_i(\mathbf{r}) = e_i\varphi_i(\mathbf{r})$$ whose eigenstates are plane waves

$$\varphi_{n_x,n_y,n_z} = \sqrt{8/L^3}\sin(k_{n_x}x/L)\sin(k_{n_y}y/L)\sin(k_{n_z}z/L) \qquad k_{n_i} = \frac{n_i\pi}{L}$$

corresponding to eigenvalues $$e_i = \frac{\hbar^2 k_i^2}{2m} = \frac{\hbar^2(k_{n_x}^2 + k_{n_y}^2 + k_{n_z}^2)}{2m} \qquad (e_0 = 0)$$

In the grand canonical ensemble the total energy and number of particles are obtained as

$$\sum_i e_i n_i = E \qquad \sum_i n_i = N \qquad n_i = \frac{1}{e^{\beta(e_i-\mu)} - 1} \qquad \mu < e_0$$

separating the ground state from the sum and turning summation to integration we obtain

$$N = \sum_i \frac{1}{e^{\beta(e_i-\mu)} - 1} = N_0 + \sum_{i\neq 0}\frac{1}{e^{\beta(e_i-\mu)} - 1} \approx N_0 + \int \frac{g(e)}{e^{\beta(e-\mu)} - 1}de \qquad \beta = \frac{1}{k_B T}$$

case study :: an ideal Bose gas in a 3D box

need to calculate the density of states (number of eigenstates per unit energy), the phase space volume is

$$G(e) = \iint \frac{dr^3 dp^3}{(2\pi\hbar)^3} = \frac{V}{(2\pi\hbar)^3} \int 4\pi p^2 dp \qquad e = \frac{\hbar^2 k^2}{2m} = \frac{p^2}{2m}$$

hence the density of states is

$$g(e) = \frac{dG(e)}{de} = \frac{V m^{3/2}}{\sqrt{2}\pi^2\hbar^3}\sqrt{e}$$

"remember" the integral $g_p(z) = \frac{1}{\Gamma(p)} \int_0^\infty x^{p-1} \frac{1}{z^{-1}e^x - 1} dx \qquad \Gamma(3/2) = \frac{\sqrt{\pi}}{2}$ using which we get

$$N = N_0 + N_{th} \approx N_0 + \int \frac{g(e)}{e^{\beta(e-\mu)} - 1} de \qquad N_{th} = \frac{V}{\lambda_{th}^3} g_{3/2}(e^{\beta\mu})$$

where $\lambda_{th} = \frac{h}{\sqrt{2\pi m k_B T}}$

Bose-Einstein condensation occurs when the population in the excited states cannot account for all particles. The maximum of N_{th} is obtained when the chemical potential becomes equal to the ground state energy

$$\mu \rightarrow e_0$$

$$\boxed{n_c \lambda_{th}^3 = g_{3/2}(1) \approx 2.615}$$

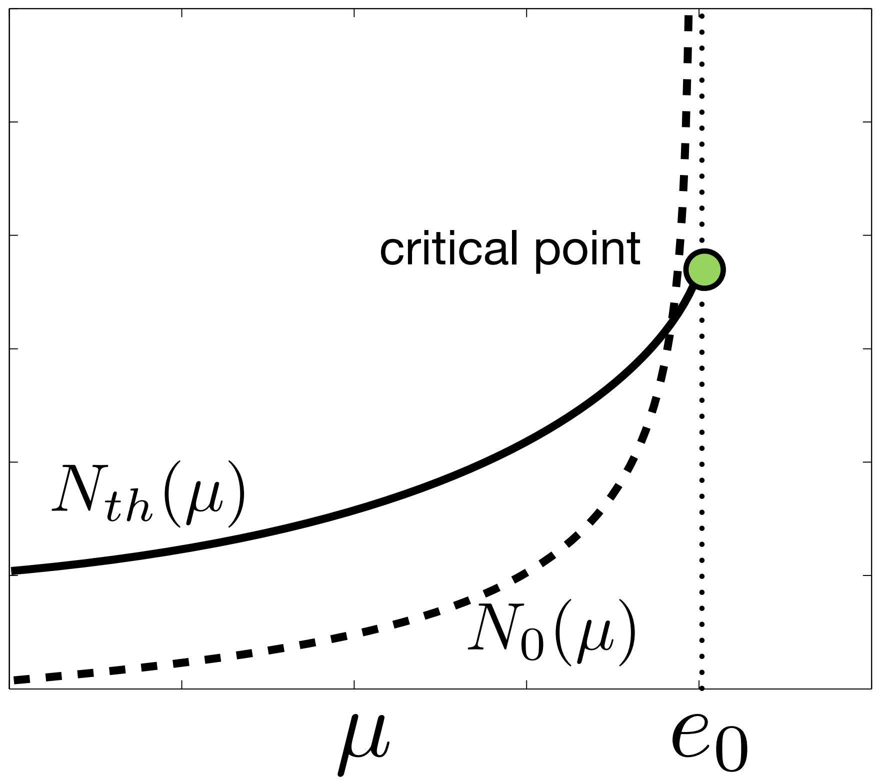
$$\boxed{T_c = \frac{2\pi\hbar^2}{g_{3/2}(1) m k_B} n^{2/3}}$$

below the critical temperature the fraction of particles in the thermal component is

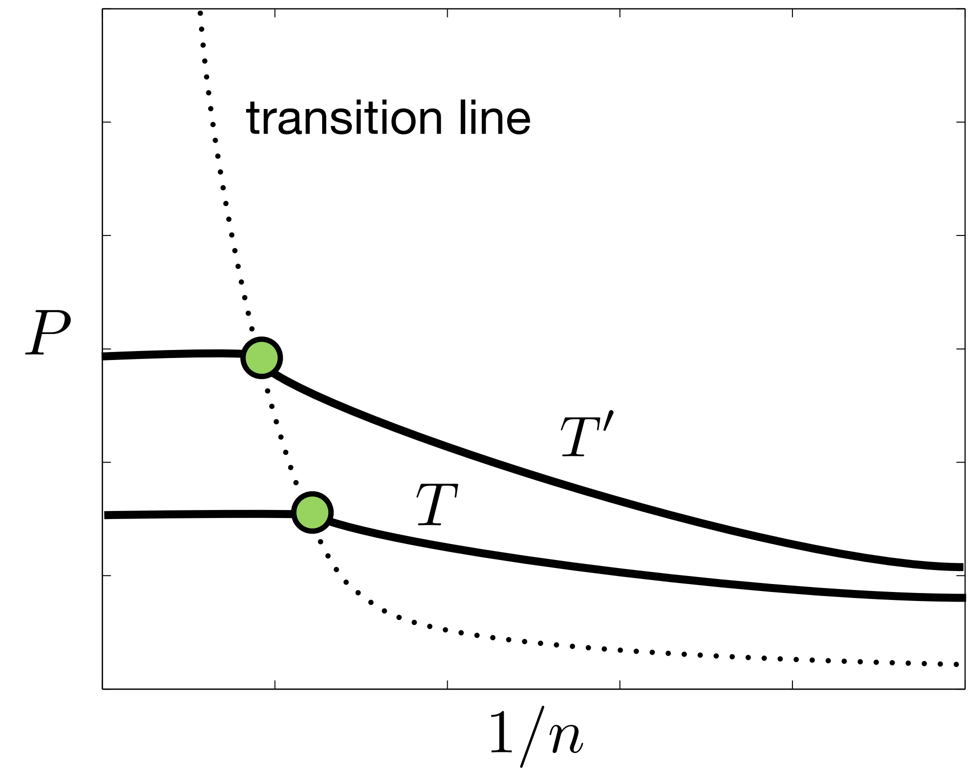
$$\frac{N_{th}}{N} = \frac{\lambda_c^3}{\lambda_{th}^3} = \frac{N - N_0}{N}$$

$$\boxed{\frac{N_0}{N} = 1 - \left(\frac{T}{T_c}\right)^{3/2}}$$

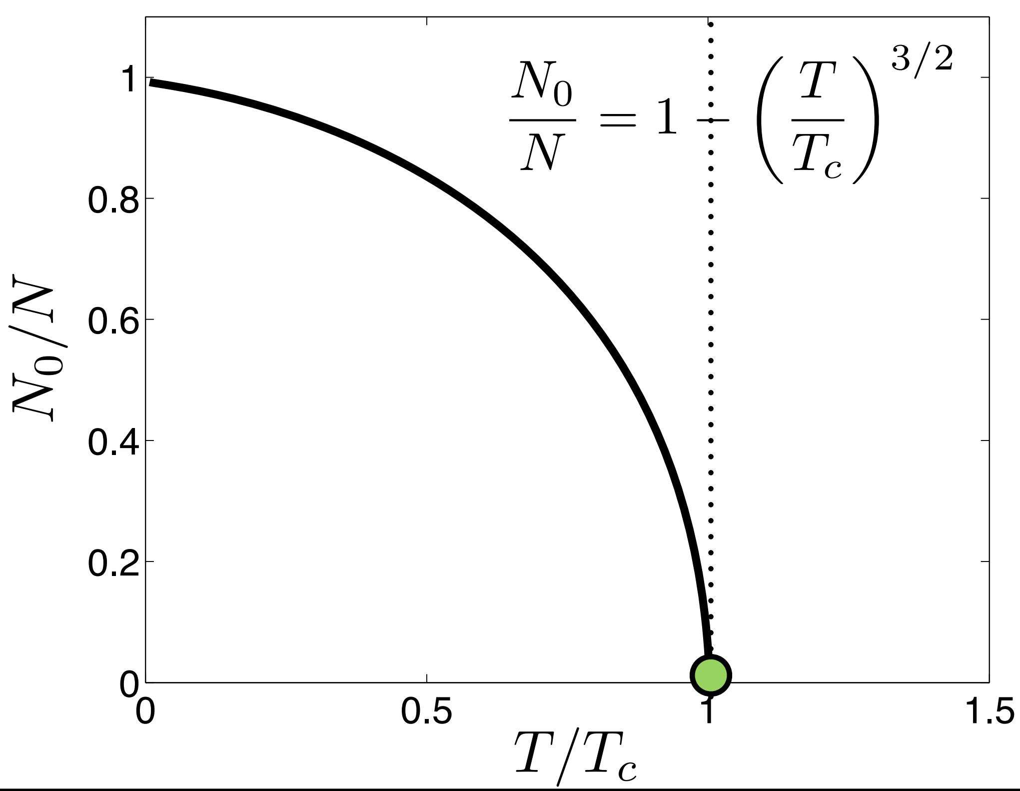
chemical potential



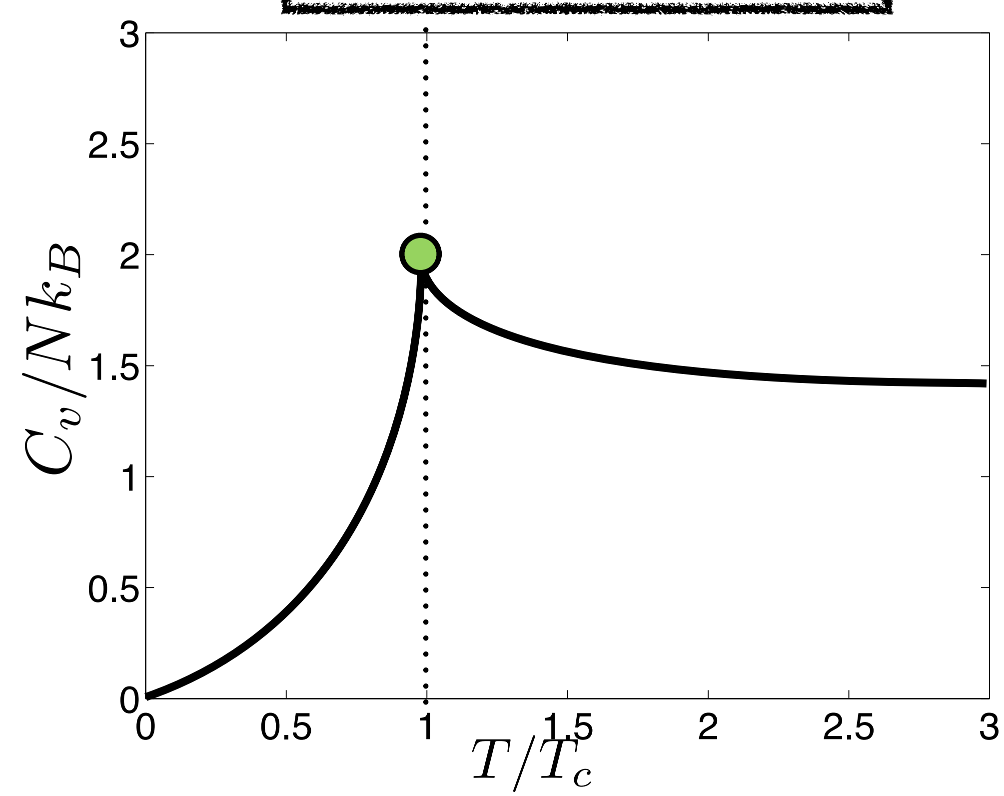
vapour pressure



condensate fraction



specific heat



case study :: an ideal Bose gas in a 3D harmonic trap

$$\left(-\frac{\hbar^2\nabla^2}{2m}+\frac{1}{2}m(\omega_x^2x^2+\omega_y^2y^2+\omega_z^2z^2)\right)\varphi(\mathbf{r})=e_i\varphi(\mathbf{r})$$

Eigenstates are now Hermite functions (polynomial times a Gaussian)

$$\phi(x)\sim H_n(x)e^{-x^2/\sigma^2}$$
$$H_0(x)=1$$

with eigenenergies $e_{n_x,n_y,n_z}=\left(\frac{1}{2}+n_x\right)\hbar\omega_x+\left(\frac{1}{2}+n_y\right)\hbar\omega_y+\left(\frac{1}{2}+n_z\right)\hbar\omega_z$

ground state wavefunction is a Gaussian whose waist is given by the harmonic oscillator length

$$\boxed{a_{ho}=\sqrt{\frac{\hbar}{m\omega}}}$$

as for free gas BEC occurs when
$\mu\to e_{0,0,0}$

$$N-N_0=\sum_{n_x,n_y,n_z\neq 0}\frac{1}{e^{\beta\hbar(\omega_xn_x+\omega_yn_y+\omega_zn_z)}-1}\approx\int_0^\infty\frac{dn_xdn_ydn_z}{e^{\beta\hbar(\omega_xn_x+\omega_yn_y+\omega_zn_z)}-1}=\zeta(3)\left(\frac{k_BT}{\hbar\omega_{ho}}\right)^3$$

Riemann zeta function $\zeta(p)=g_p(1)$

$$\omega_{ho}=(\omega_x\omega_y\omega_z)^{1/3}$$

hence the critical temperature and the condensate fraction are

$$\boxed{k_BT_c=\hbar\omega_0\left(\frac{N}{\zeta(3)}\right)^{1/3}\approx 0.94\hbar\omega_0N^{1/3}}$$

$$\boxed{\frac{N_0}{N}=1-\left(\frac{T}{T_c}\right)^3}$$

End of ideal gas - enter (weak) particle interactions

Hamiltonian has it all

$$\hat{H} = \sum_{\sigma} \int \hat{\Psi}^{\dagger}_{\sigma}(\mathbf{r}) V_{\mathrm{sp}}(\mathbf{r}) \hat{\Psi}_{\sigma}(\mathbf{r}) d\mathbf{r} + \sum_{\sigma\alpha\beta\gamma} \frac{1}{2} \iint \hat{\Psi}^{\dagger}_{\sigma}(\mathbf{r}) \hat{\Psi}^{\dagger}_{\alpha}(\mathbf{r}') V_{\mathrm{int}}(\mathbf{r}, \mathbf{r}') \hat{\Psi}_{\beta}(\mathbf{r}') \hat{\Psi}_{\gamma}(\mathbf{r}) d\mathbf{r} d\mathbf{r}'$$

$$\hat{\Psi}_{\sigma}(\mathbf{r}) \rightarrow \phi(\mathbf{r}, t)$$

at T = 0, all particles in a single many-body state

$$V_{\mathrm{int}}(\mathbf{r}, \mathbf{r}') = g\delta(\mathbf{r} - \mathbf{r}')$$

at T = 0, for a dilute gas only lowest order partial waves contribute to the scattering cross-section with g proportional to the s-wave scattering length a

$$g = \frac{4\pi\hbar^2 a}{m}$$

$$\boxed{\hat{H} = \int \phi^*(\mathbf{r}) V_{\mathrm{sp}}(\mathbf{r}) \phi(\mathbf{r}) d\mathbf{r} + \frac{g}{2} \int \phi^*(\mathbf{r}) \phi^*(\mathbf{r}) \phi(\mathbf{r}) \phi(\mathbf{r}) d\mathbf{r}}$$

$$E_{\mathrm{kin}} = \int \phi^*(\mathbf{r}, t) \left(-\frac{\hbar^2}{2m} \nabla^2 \right) \phi(\mathbf{r}, t) d\mathbf{r}$$

single particle kinetic energy

$$E_{\mathrm{pot}} = \int V_{\mathrm{ext}}(\mathbf{r}, t) |\phi(\mathbf{r}, t)|^2 d\mathbf{r}$$

single particle potential energy

$$E_{\mathrm{int}} = \frac{g}{2} \int |\phi(\mathbf{r}, t)|^4 d\mathbf{r}$$

many-body Hartree mean-field interaction energy

rigorous result for power law r^{ν} potentials: (virial theorem)

$$2E_{\mathrm{kin}} - \nu E_{\mathrm{pot}} + 3E_{\mathrm{int}} = 0$$

Gross-Pitaevskii equation: dilute scalar system at T = 0

$$\hat{H} = \int \phi^*(\mathbf{r}) V_{\rm sp}(\mathbf{r}) \phi(\mathbf{r}) d\mathbf{r} + \frac{g}{2} \int \phi^*(\mathbf{r}) \phi^*(\mathbf{r}) \phi(\mathbf{r}) \phi(\mathbf{r}) d\mathbf{r}$$

$$i\hbar \frac{\partial \phi(\mathbf{r},t)}{\partial t} = \frac{\delta \hat{H}}{\delta \phi^*(\mathbf{r},t)} = \lim_{h\to 0} \frac{H[\phi^*(\mathbf{r}) + h\delta(\mathbf{r}-\mathbf{r}')] - H[\phi^*(\mathbf{r})]}{h}$$

$$\boxed{i\hbar \frac{\partial}{\partial t}\phi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2 \phi(\mathbf{r},t) + V_{\rm ext}(\mathbf{r},t)\phi(\mathbf{r},t) + g|\phi(\mathbf{r},t)|^2 \phi(\mathbf{r},t)}$$

probability density = density of particles $|\phi(\mathbf{r},t)|^2 = n(\mathbf{r},t)$

$$\boxed{\int |\phi(\mathbf{r},t)|^2 d\mathbf{r} = N}$$

macroscopic wavefunction / order parameter

look for stationary states: $\phi(\mathbf{r},t) = \phi(\mathbf{r}) e^{-i\mu t/\hbar}$

$$\underbrace{\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\mathbf{r}) + g|\phi(\mathbf{r})|^2\right)}_{\text{GP hamiltonian}} \phi(\mathbf{r}) = \mu \phi(\mathbf{r})$$

chemical potential:
"speed of phase rotation"

chemical potential:
"an energy eigenvalue"

chemical potential:
"Lagrange multiplier"

Thomas-Fermi approximation: ignore density gradients

good approximation in bulk for large interacting condensates - always fails near the condensate surface: highly useful as it enables analytical solutions!

$$-\frac{\hbar^2}{2m}\nabla^2\phi(\mathbf{r}) + V_{\rm ext}(\mathbf{r})\phi(\mathbf{r}) + g|\phi(\mathbf{r})|^2\phi(\mathbf{r},t) = \mu\phi(\mathbf{r})$$

ignore gradient of density (but not gradients of phase!)

$$\boxed{n(\mathbf{r}) = (\mu - V_{\rm ext}(\mathbf{r}))/g}$$

harmonic trap $\quad V_{\rm ext}(\mathbf{r}) = \frac{1}{2}m\omega^2 r^2$

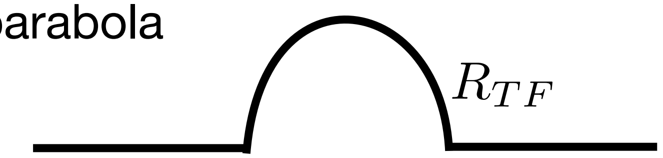
size of condensate $\quad R_{TF} = \sqrt{\frac{2\mu_{TF}}{m\omega_{ho}^2}}$

chemical potential from normalisation condition $\quad \mu_{TF} = \frac{\hbar\omega_{ho}}{2}\left(\frac{15Na}{a_{ho}}\right)^{2/5}$

energy $\quad \frac{E_{TF}}{N} = \frac{5}{7}\mu_{TF}$

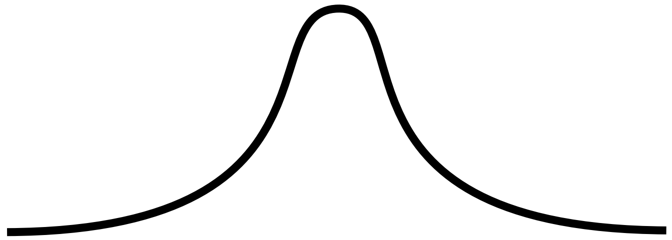
in a harmonic trap TF density is an inverted parabola

discontinuity at boundary



in uniform system $\quad V_{ext} = 0 \quad \mu = gn$

compare with the Gaussian density of non-interacting oscillator ground state



Time-of-flight: remove the trap and the condensate expands, bigger things are easier to see!

Fourier transform to obtain momentum distribution

$$\phi_0(p,t_0) = \mathcal{F}\{\phi_0(r,t_0)\}$$

free propagation in momentum space

$$\phi_0(p,t) = \phi_0(p,t_0)e^{-i\frac{p^2}{2m}t/\hbar}$$

back transform to obtain real space distribution

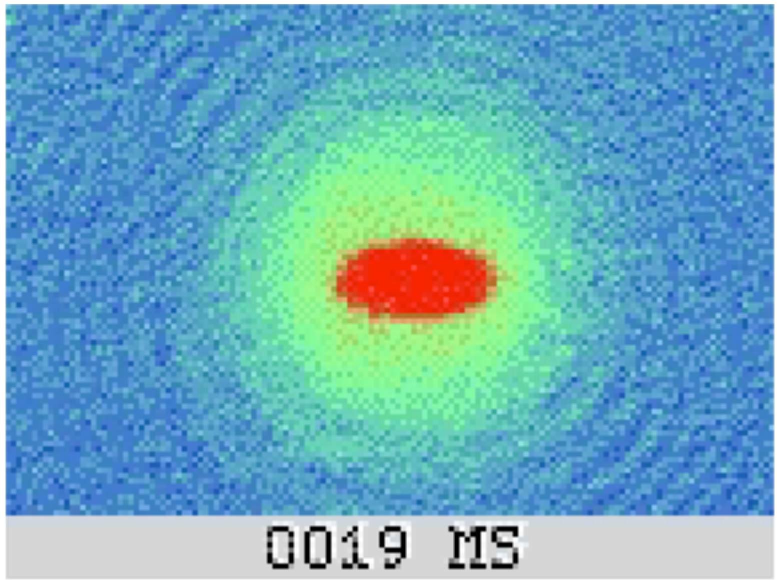
$$\phi_0(r,t) = \mathcal{F}^{-1}\{\phi_0(p,t)\}$$

for Gaussian initial distribution ignoring interactions

$$n_0(r,t) \sim e^{\frac{-r_i^2}{a_i^2(1+\omega_i^2 t^2)}}$$

$$a_i^2 = \hbar/m\omega_i$$

time-of-flight (TOF) evolution



M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.M. Kurn, D.S. Durfee, and W. Ketterle: Phys. Rev. Lett. 77, 416-419 (1996).

rule of thumb:
In-trap and TOF density profiles related by a linear scaling transformation

$$R_i(t) \sim R_i(0)\omega_i t$$

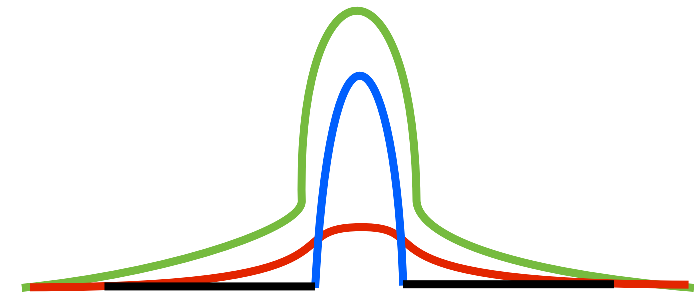
ideal gas, instead, expands isotropically

$$n_{th}(r,t) \sim e^{\frac{-m\omega_i^2 r_i^2}{2kT(1+\omega_i^2 t^2)}}$$

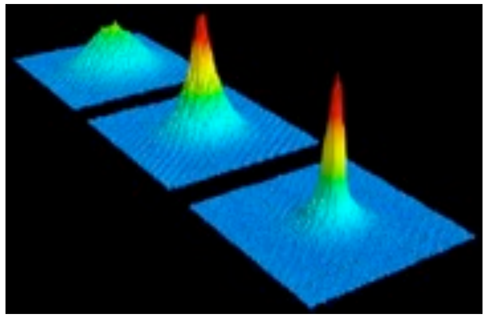
anisotropic expansion of condensate!
(narrow things in real space become wide in Fourier space and vice versa)

“Smoking guns” of a BEC or how do you prove it experimentally?

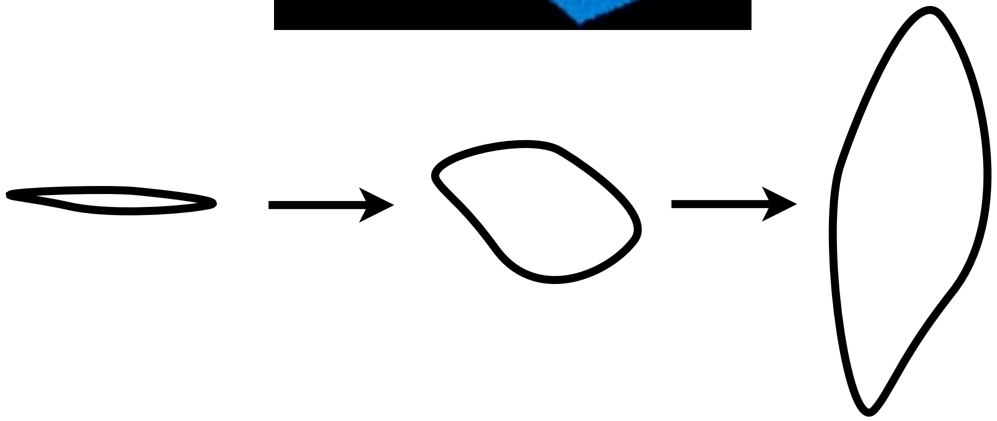
smoking gun of a BEC #1:
bimodal Gaussian + Thomas-Fermi
in-situ density distribution in a harmonic trap



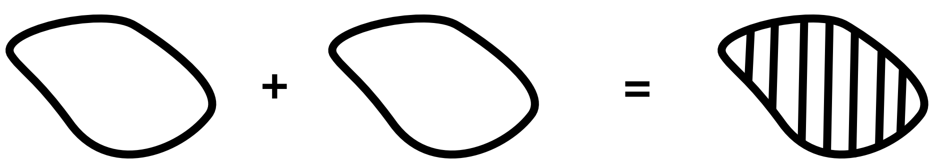
smoking gun of a BEC #2:
bimodal Gaussian + Thomas-Fermi TOF velocity
distribution after being released from a harmonic trap



smoking gun of a BEC #3:
anisotropic TOF expansion



smoking gun of a BEC #4:
coherent interference fringes



smoking gun of a BEC #5:
collective modes

Lecture 2

smoking gun of a BEC #6:
quantized vortices

Lecture 3

some orders of magnitude

scattering length	$a \sim 1\mathrm{nm}$	particle interaction range
healing length	$\xi = \frac{\hbar}{\sqrt{2mgn}} \sim 1\mu\mathrm{m}$	size of vortex core
oscillator length	$a_{ho} = \sqrt{\frac{\hbar}{m\omega}} \sim 10\mu\mathrm{m}$	size of noninteracting BEC
Thomas-Fermi length	$R_{TF} = \sqrt{\frac{2\mu}{m\omega^2}} \sim 100\mu\mathrm{m}$	size of interacting BEC
critical temperature	$T_c \sim 1\mu\mathrm{K}$	formation of BEC
trapping frequency	$\omega \sim 2\pi \times 100\mathrm{Hz}$	characteristic time 10 ms
number of BEC atoms	$N_0 \sim 10^3 - 10^5$	very small number
condensate velocity	$v \sim 1\ mm/s$	quite slow