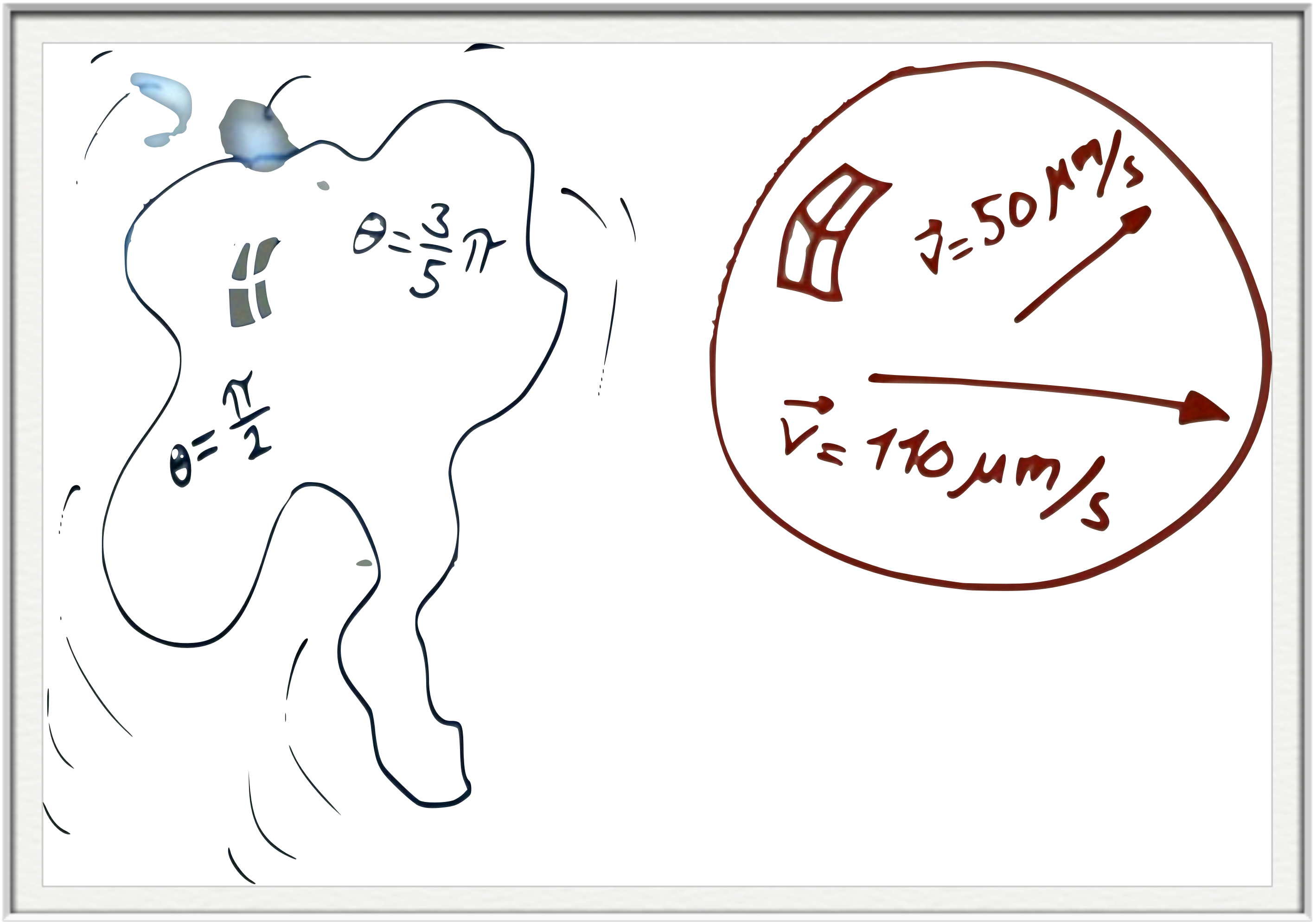
Lecture 2

where quantum liquid ventures into classical world



Detour via classical hydrodynamics

Navier-Stokes equation (Newtons law) for an incompressible fluid: $\nabla \cdot \mathbf{v} = 0$

$$\boxed{\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = \mathbf{f} - \frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{v}}$$

acceleration

forces

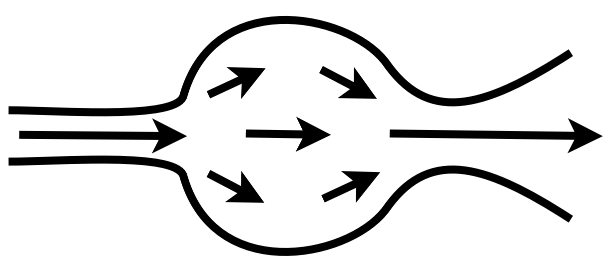
$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \frac{\partial \mathbf{v}}{\partial x_i}\frac{\partial x_i}{\partial t}$$

change of velocity field

convection

$$\boxed{\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0}$$

continuity equation
(mass conservation)



in "perfect fluid" viscosity is zero, however, it is **not superfluid** because critical velocity is zero

Quantum hydrodynamics

T = 0, ignore noncondensate, thermal, atoms

$$i\hbar\frac{\partial\phi(\mathbf{r},t)}{\partial t} = \left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}) + g|\phi(\mathbf{r},t)|^2\right)\phi(\mathbf{r},t)$$

Madelung transformation $\phi(\mathbf{r},t) = |\phi(\mathbf{r},t)|e^{i\theta(\mathbf{r},t)}$ $n_c(\mathbf{r},t) = |\phi(\mathbf{r},t)|^2$

$$\mathbf{j}(\mathbf{r},t) = \frac{\hbar}{2im}\left(\phi^*(\mathbf{r},t)\nabla\phi(\mathbf{r},t) - \phi(\mathbf{r},t)\nabla\phi^*(\mathbf{r},t)\right)$$

$$\mathbf{j}(\mathbf{r},t) \equiv mn_c\mathbf{v}_s(\mathbf{r},t) \qquad \mathbf{v}_s(\mathbf{r},t) = \frac{\hbar}{m}\nabla\theta(\mathbf{r},t)$$

superfluid velocity - **not** the velocity of a particle

complex function is zero iff both its real and imaginary parts are zero:

Im = 0: continuity equation

$$\frac{\partial n_c}{\partial t} + \nabla\cdot(n_c\mathbf{v}_s) = 0$$

together these two real valued equations are equivalent to solving the complex valued GPE!

Re = 0: Euler-like equation

$$m\frac{\partial\mathbf{v}_s}{\partial t} + \nabla\left(-\frac{\hbar^2}{2m\sqrt{n_c}}\nabla^2\sqrt{n_c} + V_{ext}(\mathbf{r},t) + gn_c + \frac{1}{2}m\mathbf{v}_s^2\right) = 0$$

irrotational inviscid classical fluid $\Leftrightarrow$ quantum liquid without quantum pressure

classical

irrotational | inviscid

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \times \cancel{(\nabla \times \mathbf{v})} - \nabla\left(\frac{\mathbf{v}^2}{2}\right) - \frac{1}{m}\nabla U - \frac{1}{mn}\nabla p + \cancel{\nu \nabla^2 \mathbf{v}}$$

$$(\mathbf{v}\cdot\nabla)\mathbf{v} = -\mathbf{v}\times(\nabla\times\mathbf{v}) + \nabla\left(\frac{\mathbf{v}^2}{2}\right)$$

identity

superfluid velocity vs particle velocity!

quantum

uniform

$$\frac{\partial \mathbf{v}}{\partial t} = \underbrace{\frac{1}{m}\nabla\left(\cancel{\frac{\hbar^2}{2m\sqrt{n_c}}\nabla^2\sqrt{n_c}}\right) - \nabla\left(\frac{\mathbf{v}^2}{2}\right)}_{\text{quantum pressure} \quad \text{momentum}} - \frac{1}{m}\nabla V_{ext}(\mathbf{r},t) - \frac{1}{mn_c}\nabla p$$

$$\mu = g n_c$$

$$dp = n_c d\mu$$

Gibbs-Duhem

in the hydrodynamic limit, expect a BEC to behave much the same way as a nonviscous, irrotational fluid

linearized quantum hydrodynamics

$$n_c = n_0 + \delta n$$

small density perturbation

$$v = \mathcal{O}(\delta n)$$

treat velocity as a perturbation

$$\frac{\partial \delta n}{\partial t} = -\nabla \cdot (n_0 \mathbf{v})$$

continuity

$$m\frac{\partial \mathbf{v}}{\partial t} = -\nabla \delta\tilde{\mu}$$

Euler equation

$$\tilde{\mu} = V + gn - \frac{\hbar^2}{2m\sqrt{n}}\nabla^2\sqrt{n}$$

$$\boxed{m\frac{\partial^2 \delta n}{\partial t^2} = \nabla \cdot (n_0 \nabla \delta\tilde{\mu})}$$

$$\delta\tilde{\mu} = \left(g + \frac{\hbar^2 q^2}{4mn_0}\right)\delta n$$

look for plane wave perturbations $\delta n \propto e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}$

to obtain a dispersion relation $m\omega^2 \delta n = \left(gnq^2 + \frac{\hbar^2 q^4}{4m}\right)\delta n$

linearized quantum hydrodynamics

$$m\omega^2 \delta n = \left(gnq^2 + \frac{\hbar^2 q^4}{4m} \right) \delta n$$

$$\boxed{\hbar\omega = \sqrt{2gnE_q + E_q^2}}$$

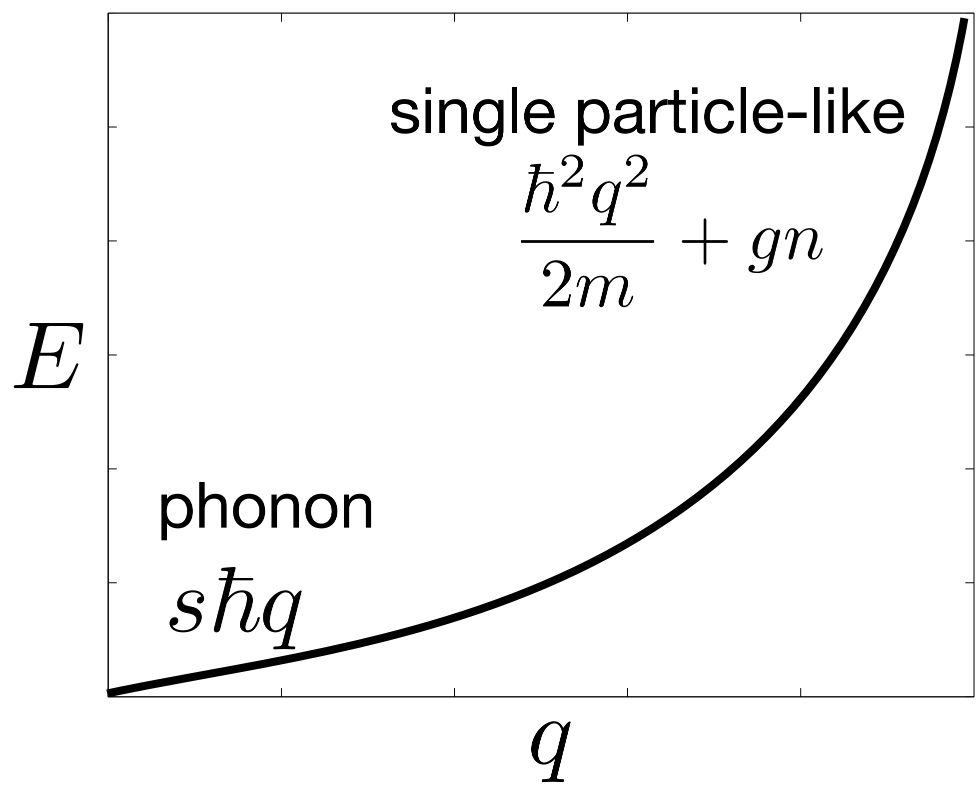
Bogoliubov dispersion relation

$$E_q = \frac{\hbar^2 q^2}{2m}$$

speed of sound

$$s = \sqrt{gn/m}$$

linear for low momenta



quadratic for high momenta

isotropic harmonic oscillator potential + TF approximation with an Ansatz

$$\delta n(\mathbf{r}) = P_\ell^{(2n_r)}(r/R) r^\ell Y_{\ell m}(\theta, \phi)$$

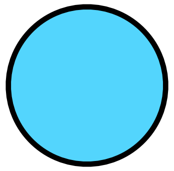
$$\boxed{\omega(n_r, \ell) = \omega_{ho}\sqrt{2n_r^2 + 2n_r\ell + 3n_r + \ell}}$$

compare: isotropic harmonic oscillator ideal-gas

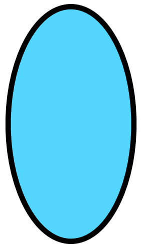
$$\omega(n_r, \ell) = \omega_{ho}(2n_r + \ell) \qquad 2n_r + \ell = n_x + n_y + n_z$$

$n_r = 0$ surface modes or surf-ons (because they live on the condensate surface)

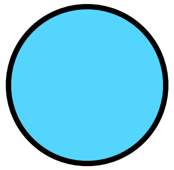
$n_r = 0, \ell = \pm 1$ dipole mode



$n_r = 0, \ell = \pm 2$ quadrupole mode



$n_r = 1, \ell = 0$ breathing mode



for large quantum numbers (surface modes) cannot throw away quantum pressure!

Landau critical velocity

Galilean transformation $E(\mathbf{v}) = E - \mathbf{p}\cdot\mathbf{v} + \frac{1}{2}Mv^2$ for condensate $E(\mathbf{v}) = E_0 + \frac{1}{2}Mv^2$

$E_p(0) = E_0 + E_p$ excitation/perturbation

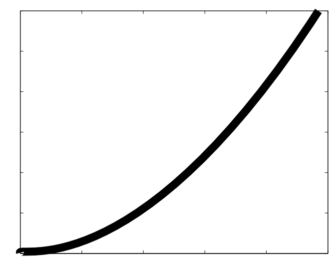
$$E_p(\mathbf{v}) = E_0 + E_p - \mathbf{p}\cdot\mathbf{v} + \frac{1}{2}Mv^2$$

superfluidity!
(need interactions)

$$\boxed{v_c = \min\left(\frac{E_p}{p}\right)}$$

ideal-gas

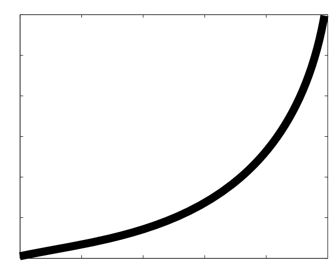
$$v_c = 0$$



$$E_p = \frac{p^2}{2m}$$

Bogoliubov spectrum:
(alkali gas BEC)

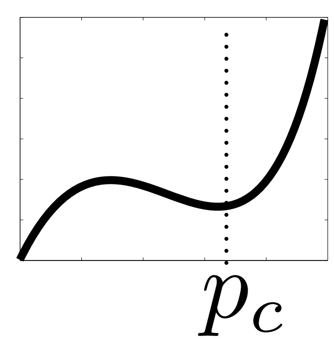
$$v_c = \sqrt{\frac{\mu}{m}} = s$$



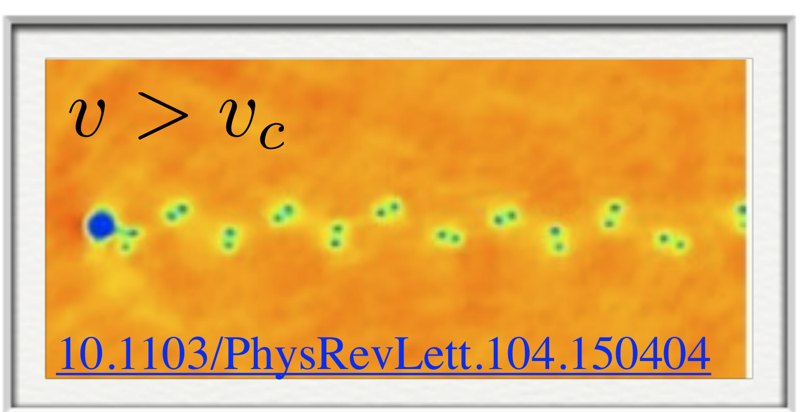
$$E_p = \sqrt{2gn\frac{p^2}{2m} + \frac{p^4}{4m^2}}$$

roton-maxon spectrum
(superfluid helium)

$$v_c = \frac{p_c}{2m}$$



$$E_p = \alpha p + \beta(p - p_c)^2$$



experimentally: ion mobility, stir with a laser spoon...

$$\boxed{\hat{H} = \sum_{\sigma} \int \hat{\Psi}^{\dagger}_{\sigma}(\mathbf{r}) V_{\mathrm{sp}}(\mathbf{r}) \hat{\Psi}_{\sigma}(\mathbf{r}) d\mathbf{r} + \sum_{\sigma\alpha\beta\gamma} \frac{1}{2} \iint \hat{\Psi}^{\dagger}_{\sigma}(\mathbf{r}) \hat{\Psi}^{\dagger}_{\alpha}(\mathbf{r}') V_{\mathrm{int}}(\mathbf{r},\mathbf{r}') \hat{\Psi}_{\beta}(\mathbf{r}') \hat{\Psi}_{\gamma}(\mathbf{r}) d\mathbf{r} d\mathbf{r}'}$$

at finite temperatures excited states play role, invoke the Bogoliubov Ansatz

$$\boxed{\hat{\Psi}_{\sigma}(\mathbf{r}) \rightarrow \phi_{\sigma}(\mathbf{r}) + \delta\hat{\psi}_{\sigma}(\mathbf{r})}$$

classical field condensate

quantum field excitations

$$\phi_{\sigma}(\mathbf{r}) \equiv \langle \hat{\Psi}_{\sigma}(\mathbf{r}) \rangle$$

spontaneous U(1) symmetry breaking

$$V_{\mathrm{int}}(\mathbf{r},\mathbf{r}') = g\delta(\mathbf{r}-\mathbf{r}')$$

weak particle interactions

reduction to a quadratic form via a mean-field approximation
(various different approximations can be made here, some better, some worse)

$$\delta\hat{\psi}^{\dagger}(\mathbf{r})\delta\hat{\psi}^{\dagger}(\mathbf{r})\delta\hat{\psi}(\mathbf{r})\delta\hat{\psi}(\mathbf{r}) \approx 4n_{th}(\mathbf{r})\delta\hat{\psi}^{\dagger}(\mathbf{r})\delta\hat{\psi}(\mathbf{r})$$

$$\delta\hat{\psi}^{\dagger}(\mathbf{r})\delta\hat{\psi}(\mathbf{r})\delta\hat{\psi}(\mathbf{r}) \approx 2n_{th}(\mathbf{r})\delta\hat{\psi}(\mathbf{r})$$

$$n_{th}(\mathbf{r}) = \langle \delta\hat{\psi}^{\dagger}(\mathbf{r})\delta\hat{\psi}(\mathbf{r}) \rangle$$

analogously for fermions, but no condensate, Wick's theorem instead of meanfield approximation and remember to use anticommutators

diagonalization with the Bogoliubov transformation to (noninteracting) quasi-particle basis

$$\boxed{\delta\hat{\psi}(\mathbf{r}) = \sum_{q} u_q(\mathbf{r})\eta_q - v^{*}_q(\mathbf{r})\eta^{\dagger}_q}$$

$$[\eta_q, \eta^{\dagger}_p] = \delta_{qp} \qquad [\eta_q, \eta_p] = 0 \qquad [\eta^{\dagger}_q, \eta^{\dagger}_p] = 0$$

$$\int u^{*}_q(\mathbf{r})u_p(\mathbf{r}) - v^{*}_q(\mathbf{r})v_p(\mathbf{r})d\mathbf{r} = \delta_{qp} \qquad \int u_q(\mathbf{r})v_p(\mathbf{r}) - u_p(\mathbf{r})v_q(\mathbf{r})d\mathbf{r} = 0$$

result: Bogoliubov-de Gennes equations

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\mathbf{r}) + g|n_c(\mathbf{r})|^2 + 2g|n_{th}(\mathbf{r})|^2\right)\phi(\mathbf{r},t) = \mu\phi(\mathbf{r})$$

GP

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\mathbf{r}) + 2gn_{tot}(\mathbf{r})\right)u_q(\mathbf{r}) + \phi(\mathbf{r})^2 v_q(\mathbf{r}) = (\mu + E_q)u_q(\mathbf{r})$$

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\mathbf{r}) + 2gn_{tot}(\mathbf{r})\right)v_q(\mathbf{r}) + \phi^*(\mathbf{r})^2 u_q(\mathbf{r}) = (\mu - E_q)v_q(\mathbf{r})$$

BdG

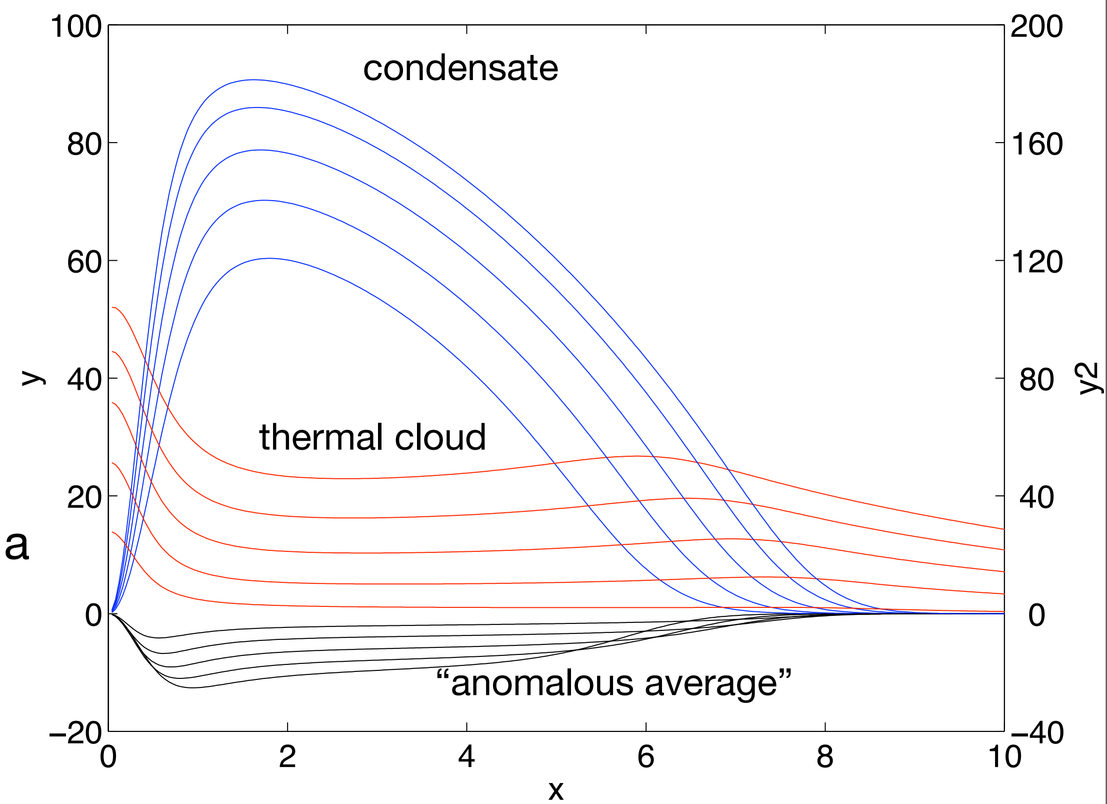
$$n_{th}(\mathbf{r}) = \sum_q f_q(|u_q(\mathbf{r})|^2 + |v_q(\mathbf{r})|^2)\boxed{+|v_q(\mathbf{r})|^2}$$

quantum depletion $\sim \sqrt{na^3}$ < 1% for weakly interacting BEC, c.f.helium

non-interacting quasi-particle sea

$$\langle \eta_q^\dagger \eta_q \rangle = f_q = \frac{1}{e^{E_q/k_B T} - 1}$$

self-consistent density profiles for a vortex state in a condensate at different temperatures



BdG for busy people: linear response of the condensate

$$i\hbar\frac{\partial}{\partial t}\phi(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\phi(\mathbf{r},t) + V_{\rm ext}(\mathbf{r},t)\phi(\mathbf{r},t) + g|\phi(\mathbf{r},t)|^2\phi(\mathbf{r},t)$$

seek for small amplitude perturbations

$$\phi(\mathbf{r},t) = [\phi_0(\mathbf{r}) + \delta\phi(\mathbf{r},t)]e^{-i\mu t/\hbar} \qquad \delta\phi(\mathbf{r},t) = \sum_q u_q(\mathbf{r})e^{-i\omega_q t} + v_q^*(\mathbf{r})e^{i\omega_q t}$$

substitute to TDGPE to obtain coupled equations:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\mathbf{r}) + 2gn_0(\mathbf{r})\right)u_q(\mathbf{r}) + \phi_0(\mathbf{r})^2 v_q(\mathbf{r}) = (\mu + E_q)u_q(\mathbf{r})$$

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\rm ext}(\mathbf{r}) + 2gn_0(\mathbf{r})\right)v_q(\mathbf{r}) + \phi_0^*(\mathbf{r})^2 u_q(\mathbf{r}) = (\mu - E_q)v_q(\mathbf{r})$$

$$E_q = \hbar\omega_q$$

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}) + 2gn_{tot}(\mathbf{r})\right)u_q(\mathbf{r}) + \phi(\mathbf{r})^2 v_q(\mathbf{r}) = (\mu + E_q)u_q(\mathbf{r})$$

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}) + 2gn_{tot}(\mathbf{r})\right)v_q(\mathbf{r}) + \phi^*(\mathbf{r})^2 u_q(\mathbf{r}) = (\mu - E_q)v_q(\mathbf{r})$$

uniform condensate $\phi = \sqrt{n_{tot}}$ plane wave Ansatz $u_q \propto e^{i(\mathbf{q}\cdot\mathbf{r})}$

or semi-classical approximation / local density approximation $-\frac{\hbar^2}{2m}\nabla^2 \rightarrow \frac{q^2}{2m} = e_q$

$$\begin{pmatrix} e_q + gn - E_q & gn \\ gn & e_q + gn + E_q \end{pmatrix} \begin{pmatrix} u_q \\ v_q \end{pmatrix} = 0$$

solve for the eigenvalues and out comes the celebrated Bogoliubov dispersion relation

$$\boxed{E_q = \sqrt{2gne_q + e_q^2}}$$

when things go unstable:

global energetic instability:

$\exists$ state with lower energy

$$\langle\phi'|H|\phi'\rangle < \langle\phi_0|H|\phi_0\rangle$$

severity: low, metastable states
may have ridiculously long lifetimes

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local energetic instability:

$\exists$ excitation with negative energy

$$E_q < 0$$

severity: low, population transfer may
be prevented by conservation laws

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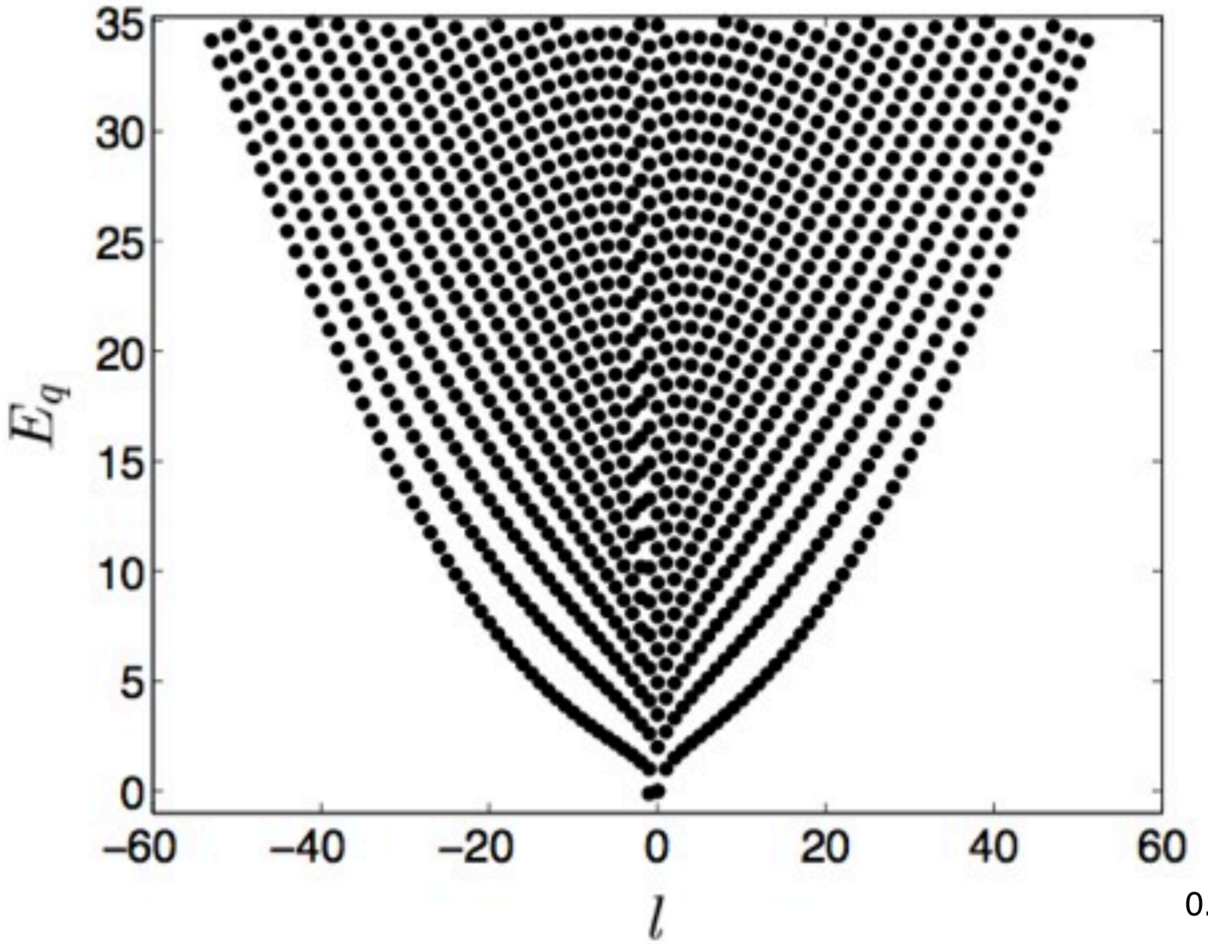
dynamic (Lyapunov) instability:

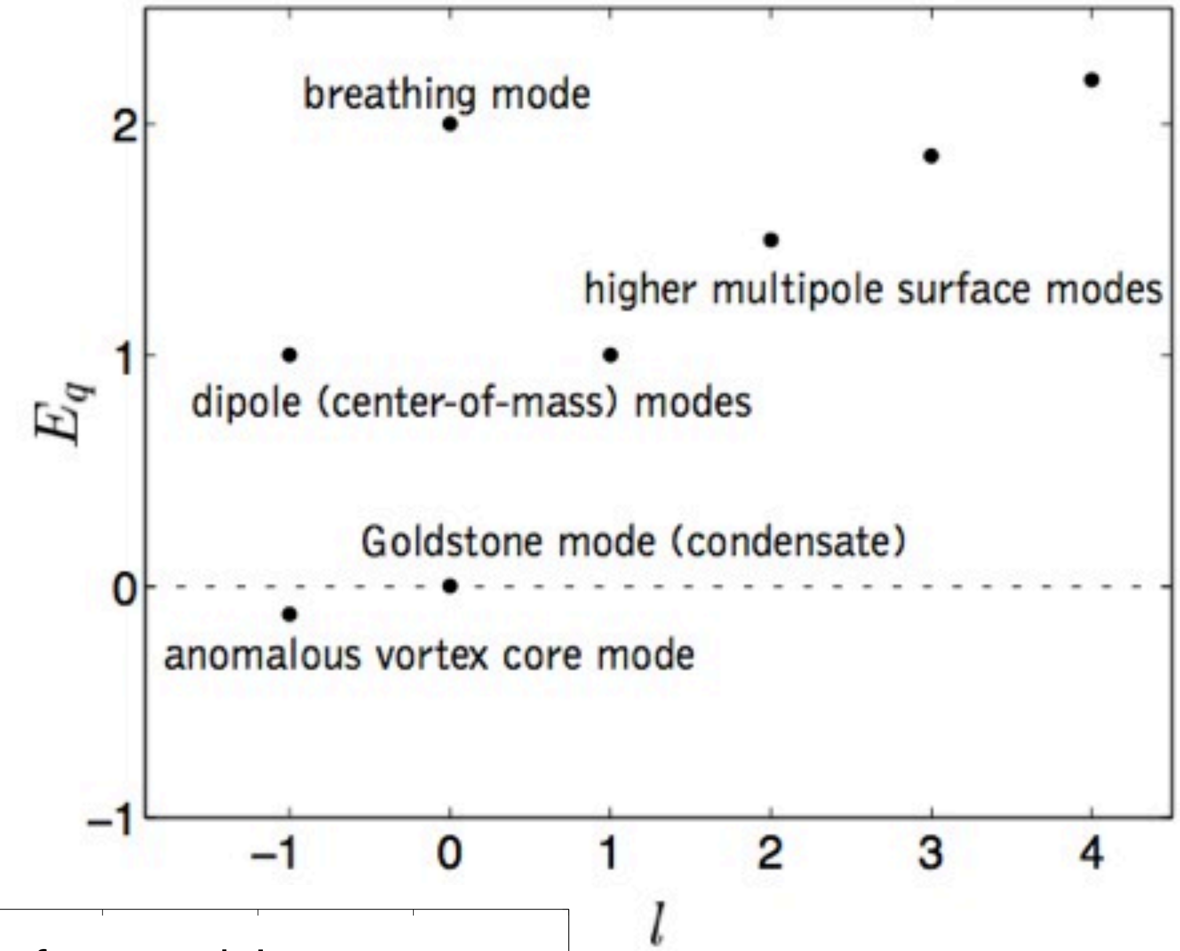
$\exists$ excitation with finite imaginary part

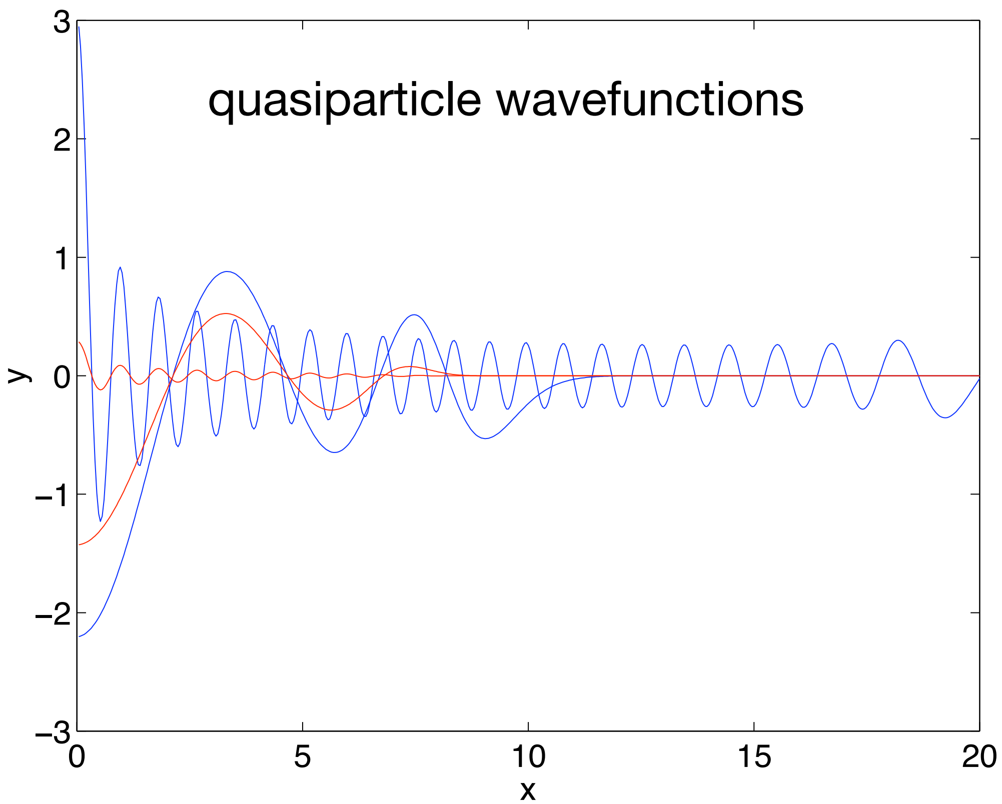
$$\mathrm{Im}(E_q) > 0$$

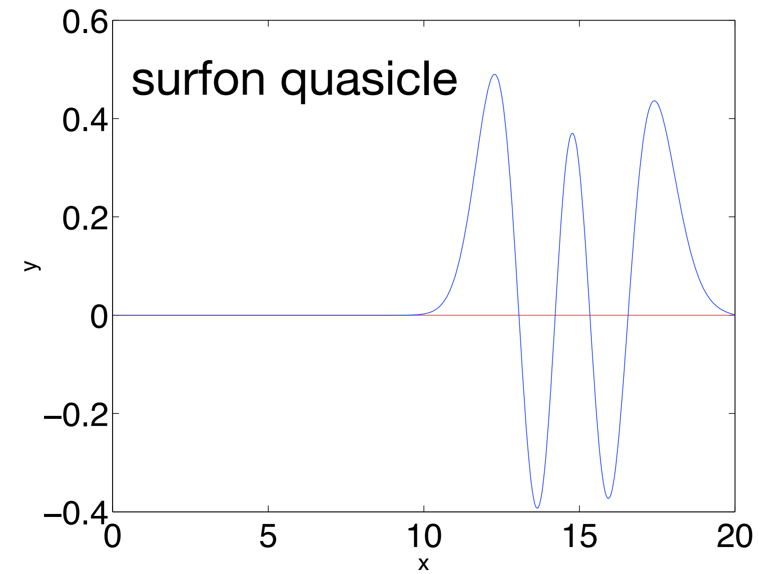
severity: high, exponentially growing
mode amplitude(s)

(low-lying modes of) Bogoliubov excitation spectrum (calculated for a vortex state)









comp hint: make sure your code reproduces accurately breathing / Kohn (centre-of-mass) modes and Goldstone boson at exactly zero energy

attractive interactions $g < 0$

$$\hbar\omega = \sqrt{2gnE_q + E_q^2} \qquad 2|g|n > \frac{\hbar^2 q_c^2}{2m}$$

Bogoliubov modes acquire imaginary part

$$q_c \sim 1/a_{ho} \qquad V \sim a_{ho}^3$$

$$N_c \propto \frac{a_{ho}}{a} \approx \frac{\mu m}{nm} = \mathcal{O}(2000)$$

maximum size of an attractive condensate

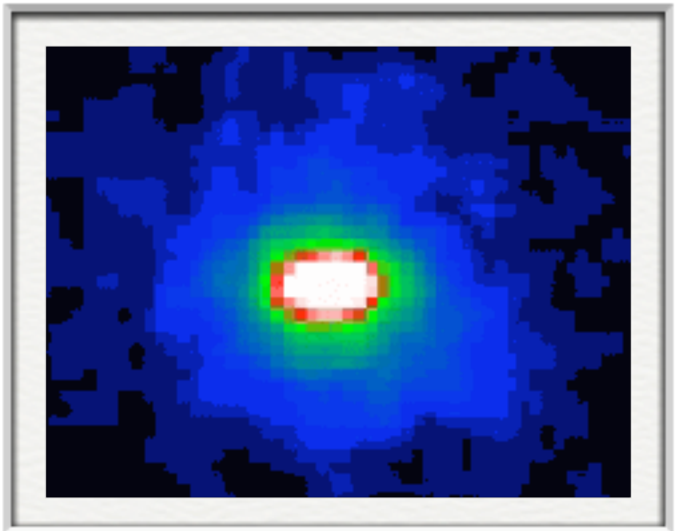
instability causes collapse when zero point pressure cannot compensate for the attraction between particles

$$e^{-i\omega_q t} \qquad e^{\pm|\omega_q|t}$$

oscillating

exponential growth /decay

in harmonically trapped rotating attractive condensate angular momentum is carried by centre-of-mass motion rather than by vortices



collapsing BEC coined a "bose nova" in analogy with a supernova (or bossanova)

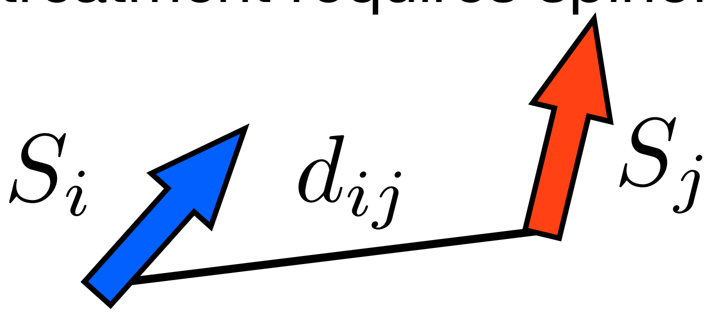
http://www.nist.gov/public_affairs/releases/bosenova.cfm

http://chandra.harvard.edu/

long-ranged inter-particle interactions

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r}) + g|\phi(\mathbf{r})|^2 + \underbrace{U(\mathbf{r},\mathbf{r}')}_{\text{nonlocal}}\right)\phi(\mathbf{r}) = \mu\phi(\mathbf{r})$$

magnetic /electric dipole-dipole force
(classical spin approximation,
full treatment requires spinor BECs)



$$U_{\text{dd}}(\mathbf{r_i},\mathbf{r_j}) = \int \frac{\overbrace{\mathbf{S_i}\cdot\mathbf{S_j}}^{\text{repulsive}} - \overbrace{\mathbf{3}(\mathbf{S_i}\cdot\mathbf{d_{ij}})(\mathbf{S_j}\cdot\mathbf{d_{ij}})}^{\text{attractive}}}{\mathbf{d^3_{ij}}}|\phi(\mathbf{r}')|^2\mathbf{d^3r}'$$

Rep. Prog. Phys. **72** 126401 (2009)

real / artificial gravity

Phys. Rev. Lett. 84, 5687 (2000)

$$U_{\text{G}}(\mathbf{r}) = -\mathbf{g_G}\int\frac{|\phi(\mathbf{r}')|^2}{|\mathbf{r}-\mathbf{r}'|}\mathbf{d^3r}'$$

self-trapped Bose-Einstein condensate!

a “boson star”