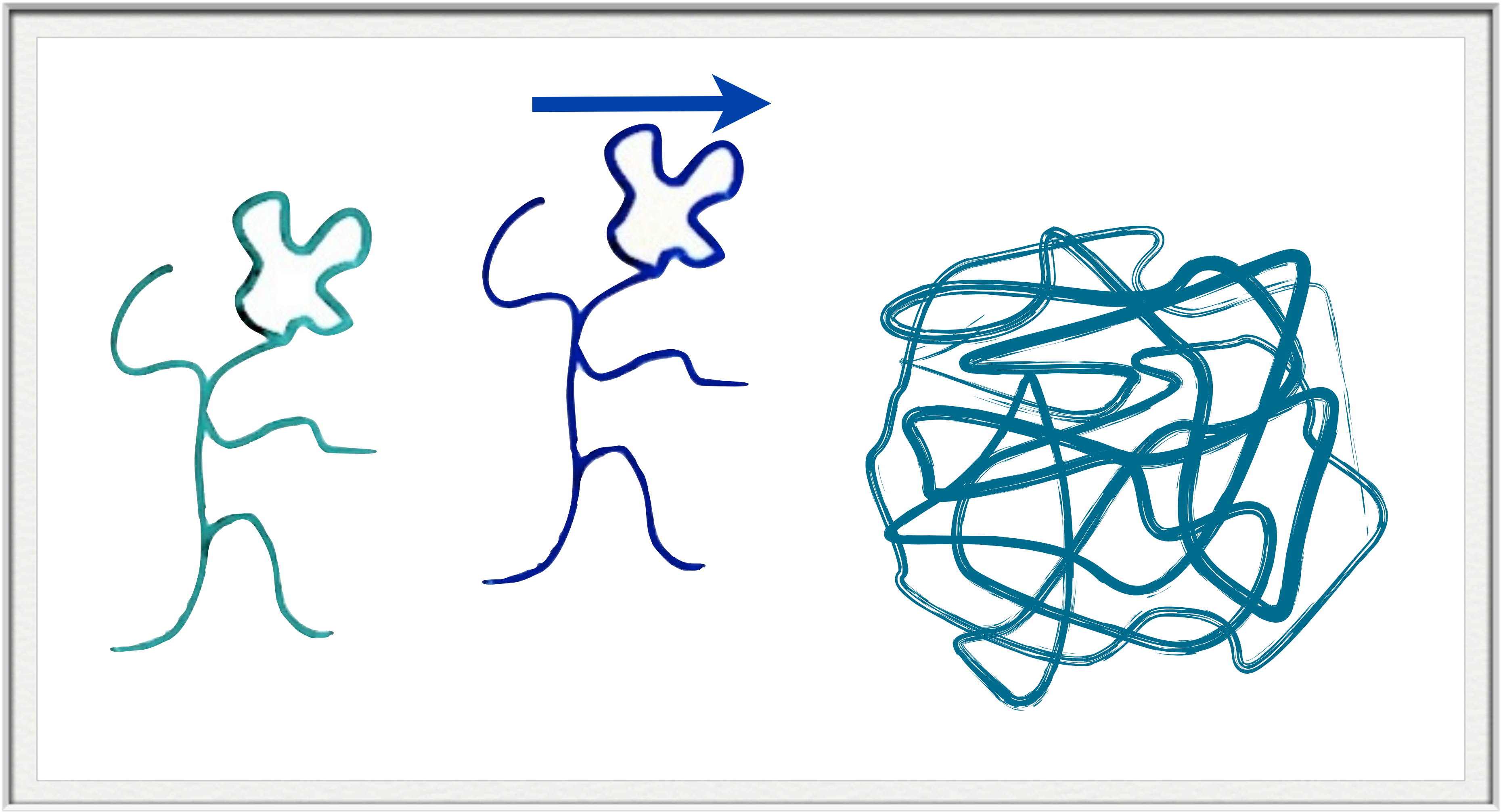
Lecture 4

where particles acquire spin and encounter turbulence



Gross-Pitaevskii equation: two-components

$$i\hbar\frac{\partial}{\partial t}\phi_1(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\phi_1(\mathbf{r},t) + V_{\text{ext}}^1(\mathbf{r},t)\phi_1(\mathbf{r},t) + g_{11}|\phi_1(\mathbf{r},t)|^2\phi_1(\mathbf{r},t) + g_{12}|\phi_2(\mathbf{r},t)|^2\phi_1(\mathbf{r},t)$$

inter-component coupling
via density potential

$$i\hbar\frac{\partial}{\partial t}\phi_2(\mathbf{r},t) = -\frac{\hbar^2}{2m}\nabla^2\phi_2(\mathbf{r},t) + V_{\text{ext}}^2(\mathbf{r},t)\phi_2(\mathbf{r},t) + g_{22}|\phi_2(\mathbf{r},t)|^2\phi_2(\mathbf{r},t) + g_{21}|\phi_1(\mathbf{r},t)|^2\phi_2(\mathbf{r},t)$$

2 x scalar BEC: ● pseudo-spin representation $\begin{pmatrix} \phi_1(\mathbf{r},t) \\ \phi_2(\mathbf{r},t) \end{pmatrix}$

two different atomic species with different masses and atomic level structure (collisions cannot convert one type of atom to another)

or

same atomic species, two different hyperfine states (decoupled from other internal states)

generalises to N component systems

Gross-Pitaevskii equation: spinor BEC

2F+1 hyperfine levels
(internally coupled wavefunction components)

F+1 coupling constants (Bose symmetry)

$F = 1$ $m_F = 1$ $m_F = 0$ $m_F = -1$

external B fields lift degeneracy, use
optical potentials to confine atoms

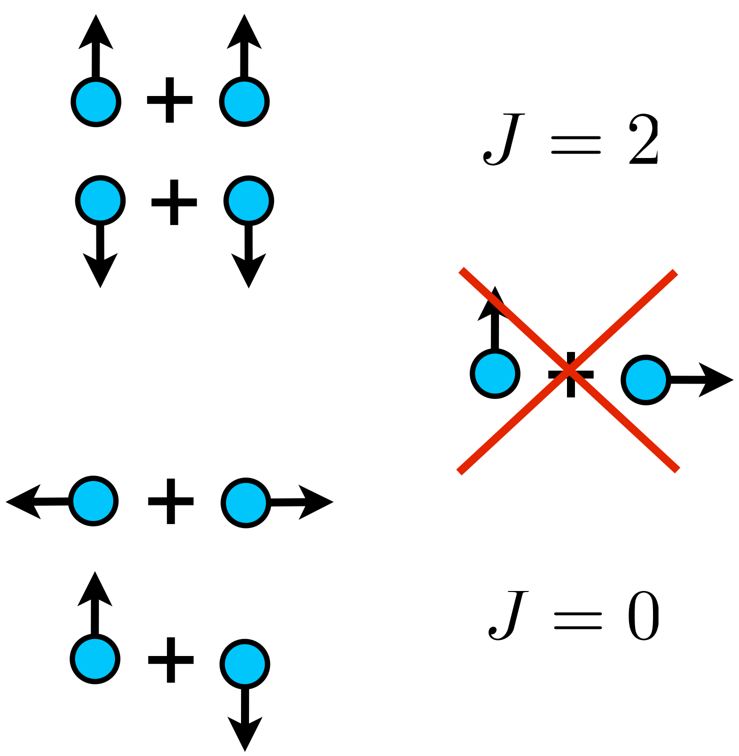
$$V_{\text{int}}(\mathbf{r}, \mathbf{r}') = \delta(\mathbf{r} - \mathbf{r}')(g_0 P_0 + g_2 P_2)$$

$$\mathbf{F_1} \cdot \mathbf{F_2} = \frac{1}{2}[J(J+1) - F_1(F_1+1) - F_2(F_2+1)]$$

$$\mathbf{F_1} \cdot \mathbf{F_2}(J=0) = -2 \qquad \mathbf{F_1} \cdot \mathbf{F_2}(J=2) = 1$$

$$P_0 = (1 - \mathbf{F_1} \cdot \mathbf{F_2})/3 \qquad P_2 = (2 + \mathbf{F_1} \cdot \mathbf{F_2})/3$$

collision of two F = 1 bosons

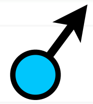


conservation of angular momentum + symmetry

$$\frac{g_0 + 2g_2}{3}\delta(\mathbf{r} - \mathbf{r}') \qquad \frac{g_2 - g_0}{3}\mathbf{F_1} \cdot \mathbf{F_2}$$

contact   spin-spin

scalar BEC: 

spinor BEC: 

Gross-Pitaevskii equation: spinor BEC

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = H\psi(\mathbf{r},t)$$

$$H = -\frac{\hbar^2}{2m}\nabla^2 + V_{\text{ext}}(\mathbf{r},t) + g_n\psi^\dagger(\mathbf{r},t)\psi(\mathbf{r},t) + g_s\psi^\dagger(\mathbf{r},t)\mathbf{F}\psi(\mathbf{r},t)\cdot\mathbf{F}$$

$$\psi(\mathbf{r},t) = \begin{pmatrix} \phi_1(\mathbf{r},t) \\ \phi_0(\mathbf{r},t) \\ \phi_-1(\mathbf{r},t) \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

vector / “spinor” order parameter

plus linear and quadratic Zeeman effects if external magnetic fields are present $-\mu_{\mathbf{B}}\cdot\mathbf{B}$

and possibly long-range $1/r^3$ dipole-dipole interactions

$$F_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \qquad F_y = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \qquad F_z = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

transformation of a spinor

$$\psi'(\mathbf{r},t) = e^{-i\gamma} e^{i\alpha\mathbf{n}\cdot\mathbf{F}} \psi(\mathbf{r},t)$$

$$\psi'(\mathbf{r},t) = e^{-i\gamma} e^{i\alpha F_z} \psi(\mathbf{r},t)$$

for the sake of simplicity we restrict to considering axisymmetric states

$$e^{i\alpha F_z} = \begin{pmatrix} e^{i\alpha} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-i\alpha} \end{pmatrix}$$

$e^{-i\gamma}$	gauge transformation	relates to orbital angular momentum
$e^{i\alpha F_z}$	spin rotation	relates to spin angular momentum

geometric phase of a single quantum vortex

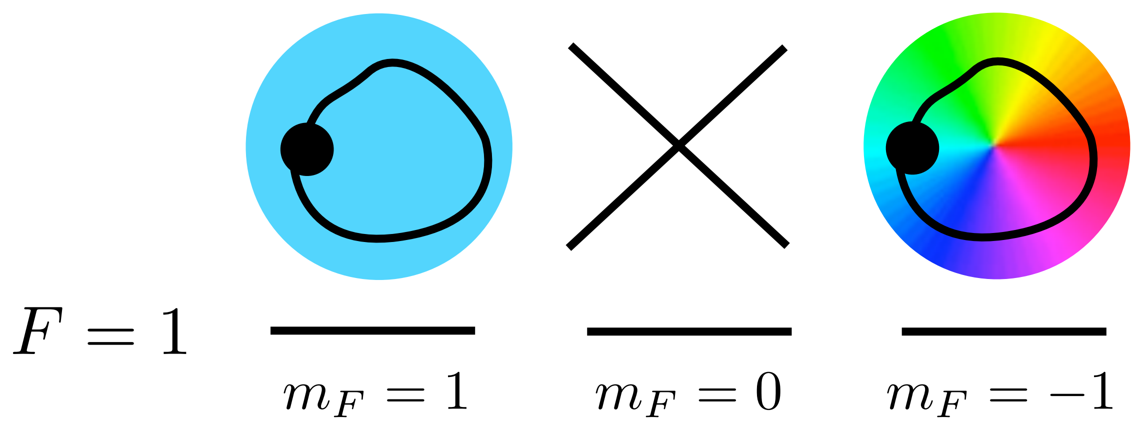
$$\psi = e^{i\theta}$$

$$\psi' = e^{-i\gamma}e^{i\theta}$$

$e^{2\pi}$
e^{0}

$$\gamma = 2\pi$$

few axisymmetric vortex types in an F = 1 spinor BEC



$$\begin{pmatrix} 1 \\ 0 \\ e^{i\theta} \end{pmatrix}$$

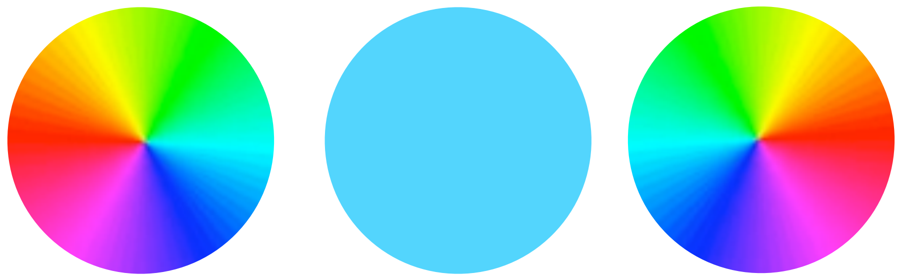
half-quantum “coreless” vortex (aka Alice-string)

$$S = 1/2 \qquad \alpha = 1/2 \times 2\pi$$

$$L = 1/2 \qquad \gamma = 1/2 \times 2\pi$$

$$e^{-i\gamma} e^{i\alpha F_z} = \begin{pmatrix} e^{i(\alpha-\gamma)} & 0 & 0 \\ 0 & e^{-i\gamma} & 0 \\ 0 & 0 & e^{-i(\alpha+\gamma)} \end{pmatrix}$$

few axisymmetric vortex types in an F = 1 spinor BEC



$F = 1$ $\quad m_F = 1 \quad m_F = 0 \quad m_F = -1$

$$\begin{pmatrix} e^{-i\theta} \\ 1 \\ e^{i\theta} \end{pmatrix}$$

polar core / pure spin vortex

$$S = 1$$

$$L = 0$$

$$\alpha = 2\pi$$

$$\gamma = 0$$

$$e^{-i\gamma} e^{i\alpha F_z} = \begin{pmatrix} e^{i(\alpha-\gamma)} & 0 & 0 \\ 0 & e^{-i\gamma} & 0 \\ 0 & 0 & e^{-i(\alpha+\gamma)} \end{pmatrix}$$

few axisymmetric vortex types in an F = 1 spinor BEC

$F = 1$ $m_F = 1$ $m_F = 0$ $m_F = -1$

$$\begin{pmatrix} 1 \\ e^{i\theta} \\ e^{i2\theta} \end{pmatrix}$$

Mermin-Ho vortex / Anderson-Toulouse vortex / skyrmion

$$\alpha = -2\pi$$

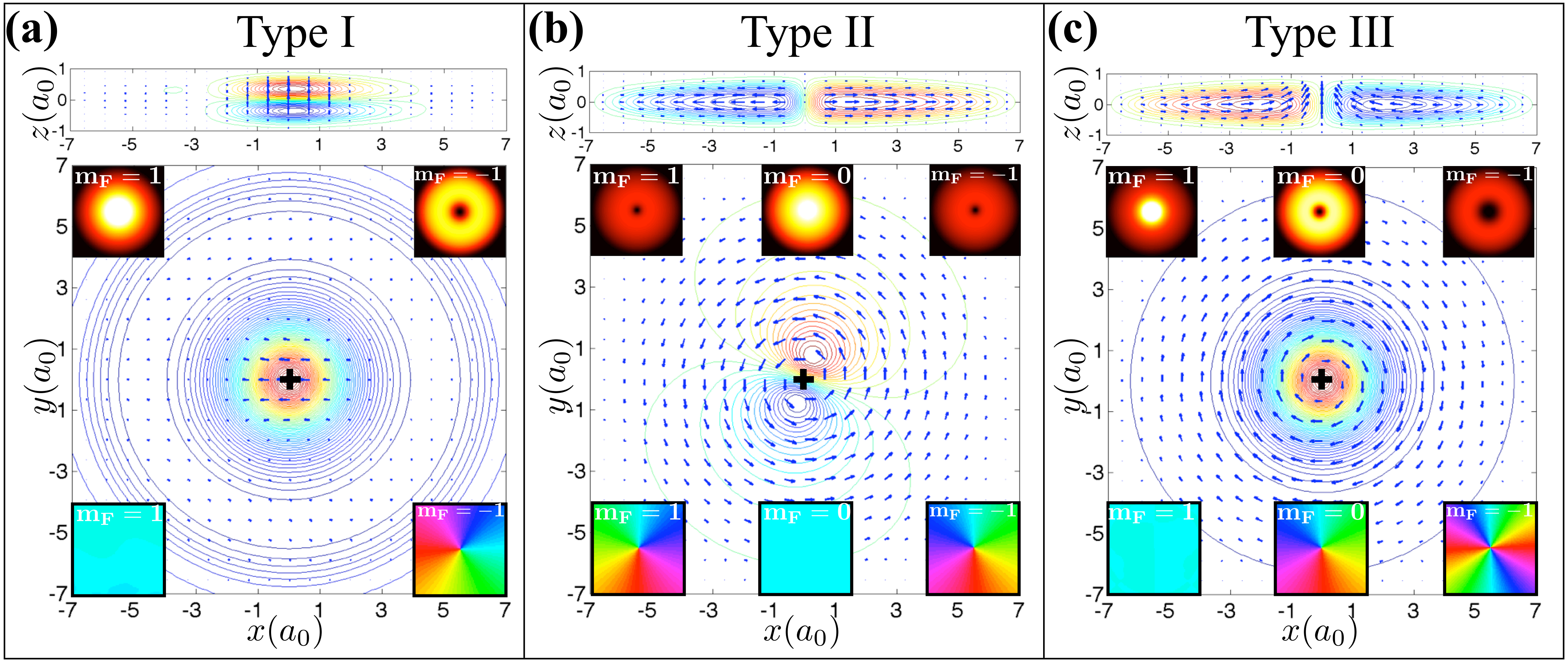
$$S = 1$$

$$\gamma = -2\pi$$

$$L = 1$$

$$e^{-i\gamma} e^{i\alpha F_z} = \begin{pmatrix} e^{i(\alpha-\gamma)} & 0 & 0 \\ 0 & e^{-i\gamma} & 0 \\ 0 & 0 & e^{-i(\alpha+\gamma)} \end{pmatrix}$$

few axisymmetric vortex types in an F = 1 spinor BEC



$S = 1/2$ $\quad$ $S = 1$ $\quad$ $S = 1$

$L = 1/2$ $\quad$ $L = 0$ $\quad$ $L = 1$

F = 2 fractional vortices some of which are “non-Abelian”

$$\begin{pmatrix} e^{i\theta} \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

1/3 vortex

$$\alpha = -1/3 \times 2\pi$$
$$\gamma = 1/3 \times 2\pi$$

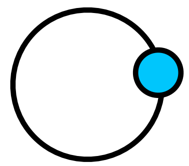
$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ e^{i\theta} \\ 0 \end{pmatrix}$$

2/3 vortex

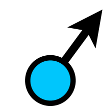
$$\alpha = 1/3 \times 2\pi$$
$$\gamma = 2/3 \times 2\pi$$

$$e^{-i\gamma}e^{i\alpha F_z} = \begin{pmatrix} e^{i(2\alpha-\gamma)} & 0 & 0 & 0 & 0 \\ 0 & e^{i(\alpha-\gamma)} & 0 & 0 & 0 \\ 0 & 0 & e^{-i\gamma} & 0 & 0 \\ 0 & 0 & 0 & e^{-i(\alpha+\gamma)} & 0 \\ 0 & 0 & 0 & 0 & e^{-i(2\alpha+\gamma)} \end{pmatrix}$$

exotic excitations in spinor systems



$$\hat{J} = \hat{L} + \hat{S}$$



orbital angular momentum:
mass currents (gauge vortices)

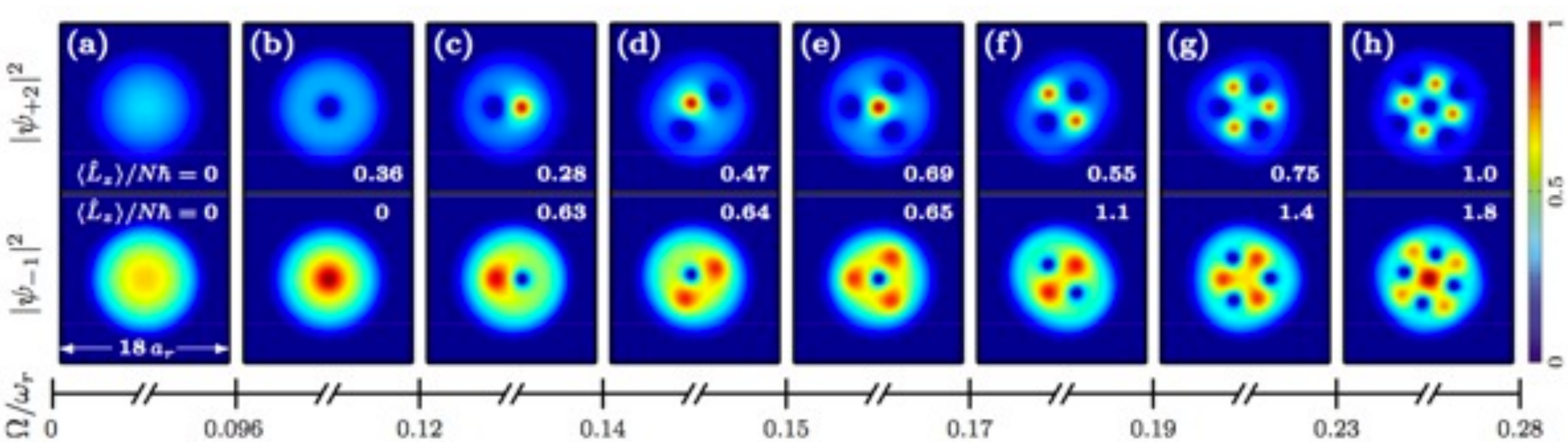
spin angular momentum:
spin currents (spin vortices)

interconversion!

magnons (spin-wave excitations)

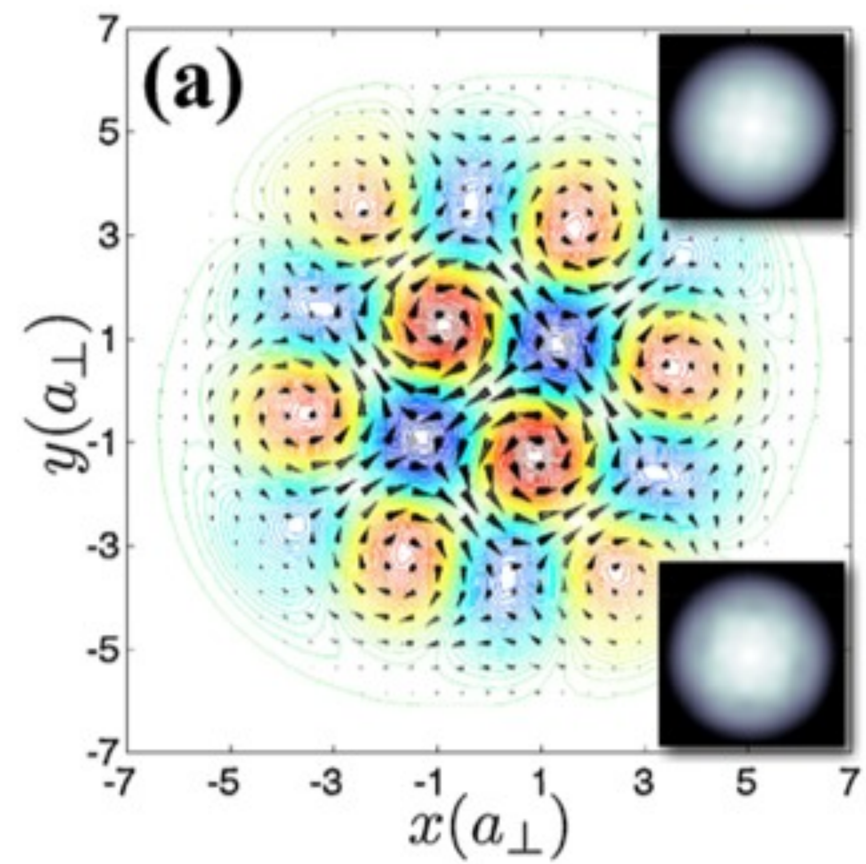
skyrmions, spin-textures, monopoles

fractional and non-Abelian vortices...



PRA **80**, 051601 (2009)

square lattice vs triangular!

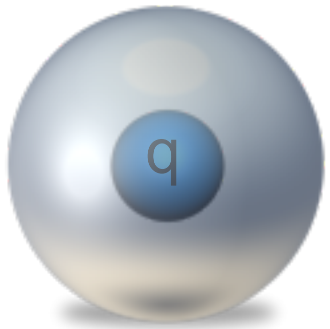


vortex-electromagnetism and (synthetic) gauge fields

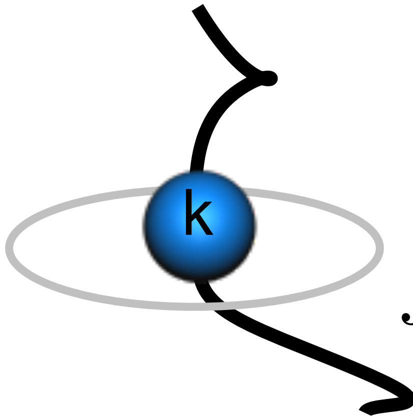
charge

vortex

3D



$$\oint E \cdot dS = q$$

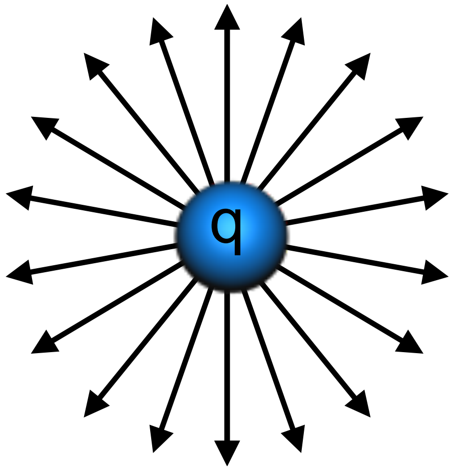


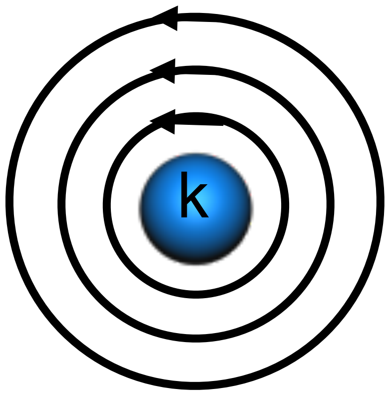
$$\oint v \cdot dl = k$$

$$U_{\mathrm{Coulomb}} \sim 1/|r - r'|$$

$$U_{\mathrm{vortex}} \sim \ln|r - r'|$$

2D





$$U_{\mathrm{Coulomb}} \sim \ln|r - r'|$$

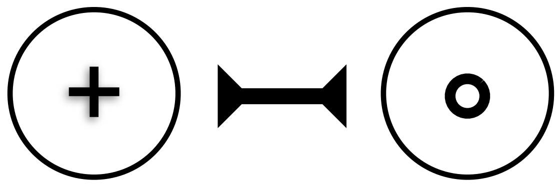
$$U_{\mathrm{vortex}} \sim \ln|r - r'|$$

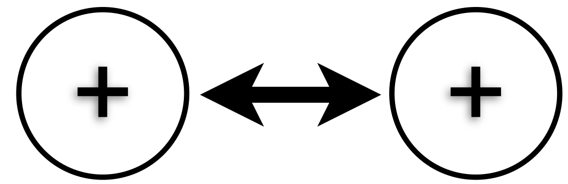
attraction vs repulsion c.f. electric charges

$$E_{+-} = A \ln\left(\frac{d}{r_c}\right)$$

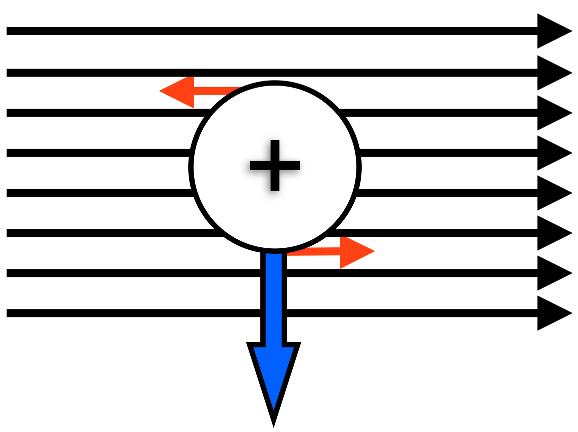
$$F = -\nabla E$$

$$E_{++} = A \ln\left(\frac{R^2}{r_c d}\right)$$

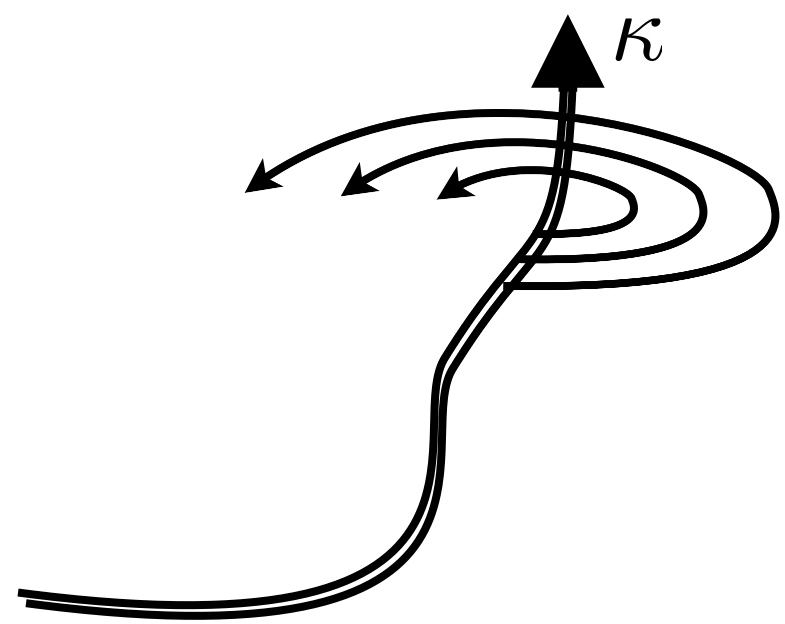




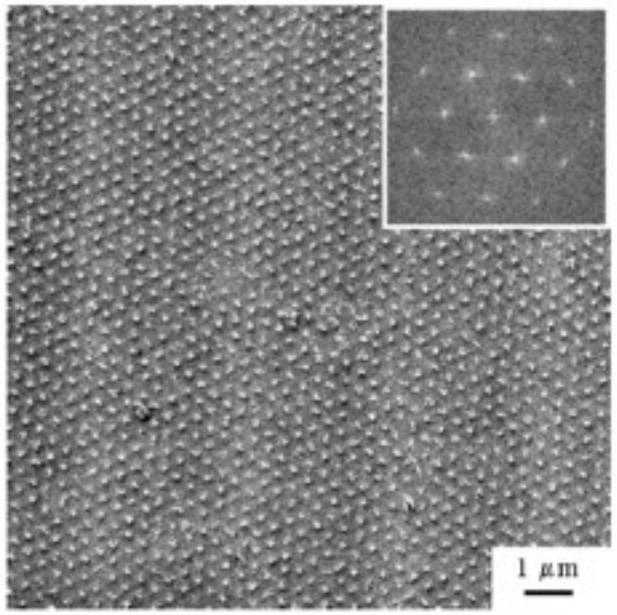
Magnus force $\qquad F_M = n\mathbf{v} \times \kappa$



c.f. current carrying wire: Biot-Savart law



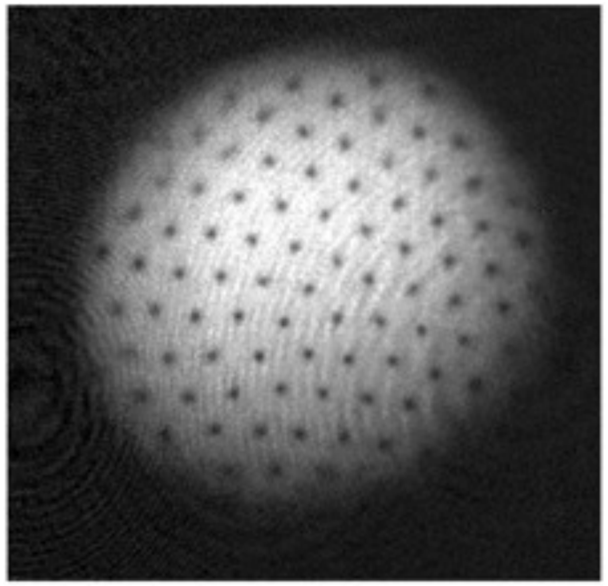
superconductor



B

BEC

Ω



$$\mathbf{G} = \nabla \times \mathbf{A}_G$$

$$H = (\mathbf{p} - q\mathbf{A}_B)^2/2m$$

$$\mathbf{A}_B = B\mathbf{x}$$

Landau gauge

$$H = p_x^2/2m + \frac{1}{2}m\omega_c^2\left(x - \frac{\hbar k_y}{m\omega_c}\right)^2$$

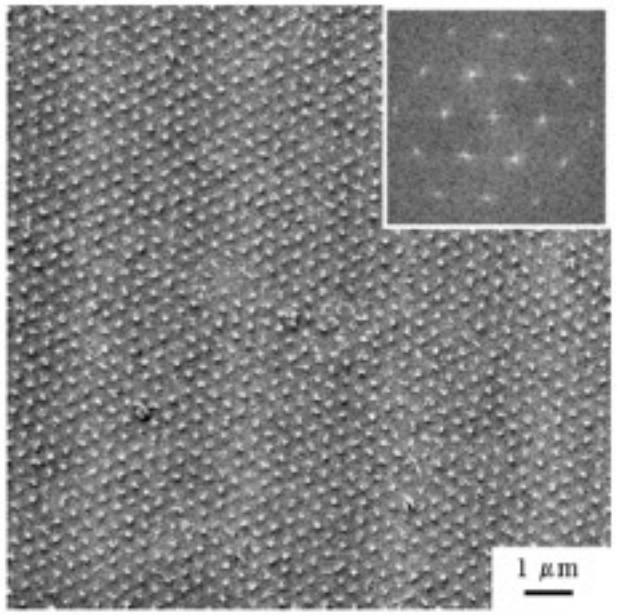
$$H = \mathbf{p}^2/2m - \boldsymbol{\Omega}\cdot\mathbf{r}\times\mathbf{p}$$

$$\mathbf{A}_\Omega = \frac{\sqrt{\gamma}}{\kappa}m\Omega(-y\mathbf{e}_x + x\mathbf{e}_y)$$

symmetric gauge

$$H = (\mathbf{p}_v - \kappa\mathbf{A}_\Omega)^2/2m_v - m\Omega^2 r_\perp^2/2$$

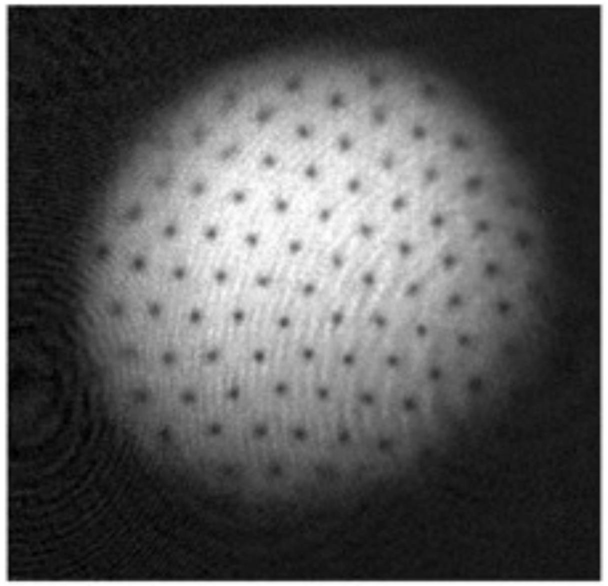
superconductor



B

BEC

Ω



$$H = (\mathbf{p} - q\mathbf{A}_B)^2/2m$$

$$H = \mathbf{p}^2/2m - \boldsymbol{\Omega} \cdot \mathbf{r} \times \mathbf{p}$$

$$H = p_x^2/2m + \frac{1}{2}m\omega_c^2\left(x - \frac{\hbar k_y}{m\omega_c}\right)^2$$

$$H = (\mathbf{p}_v - \kappa\mathbf{A}_\Omega)^2/2m_v - m\Omega^2 r_\perp^2/2 + m\omega_\perp^2 r_\perp^2/2$$

$$E_n = \hbar\omega_c\left(n + \frac{1}{2}\right)$$

Landau levels

integer and fractional quantum-Hall effects!

Synthetic magnetic fields for ultracold neutral atoms

Y.-J. Lin, R. L. Compton, K. Jiménez-García, J. V. Porto & I. B. Spielman

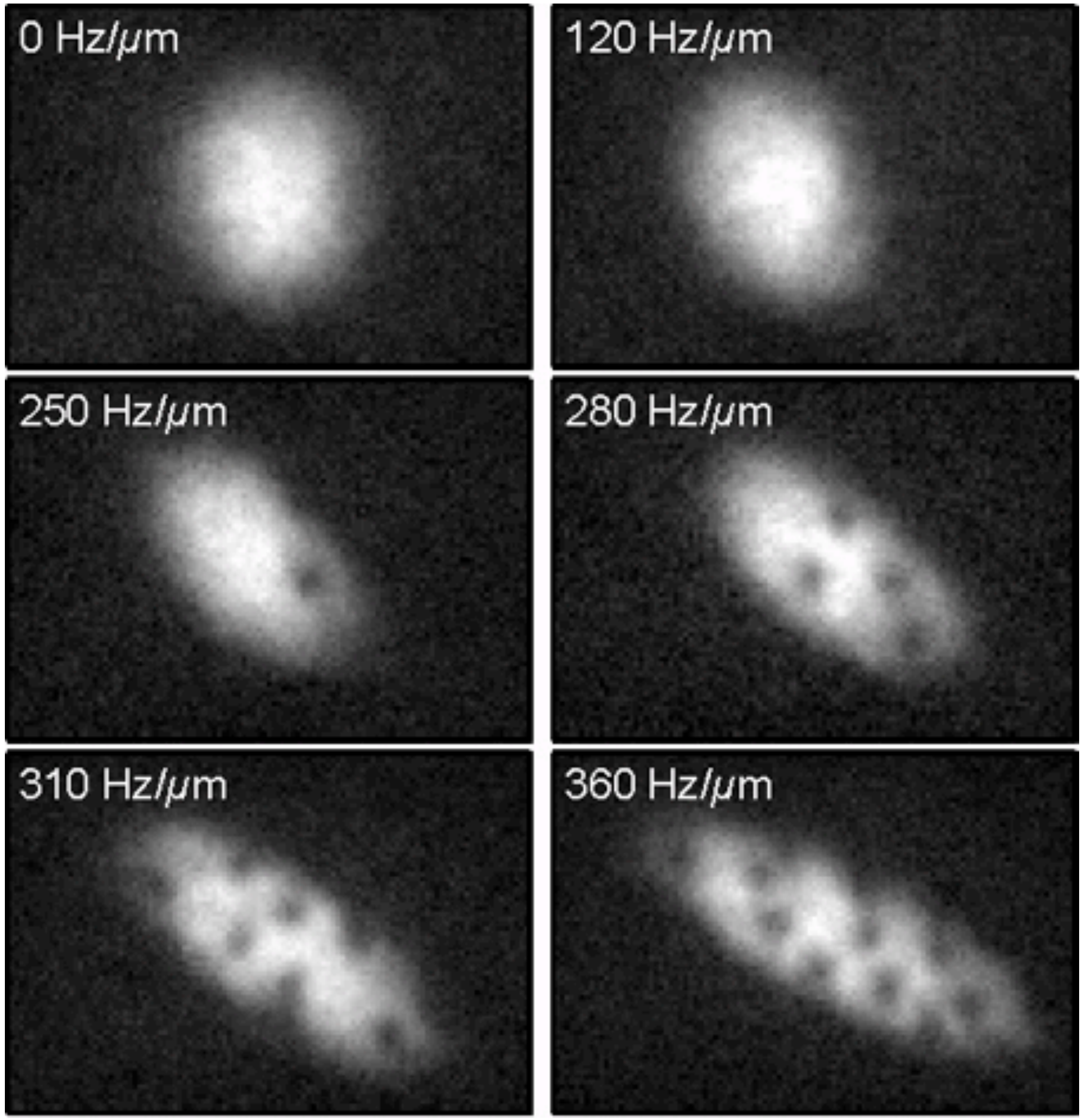
Nature **462**, 628-632 (2009)

effective vector potential created by tuning the energy-momentum dispersion relation

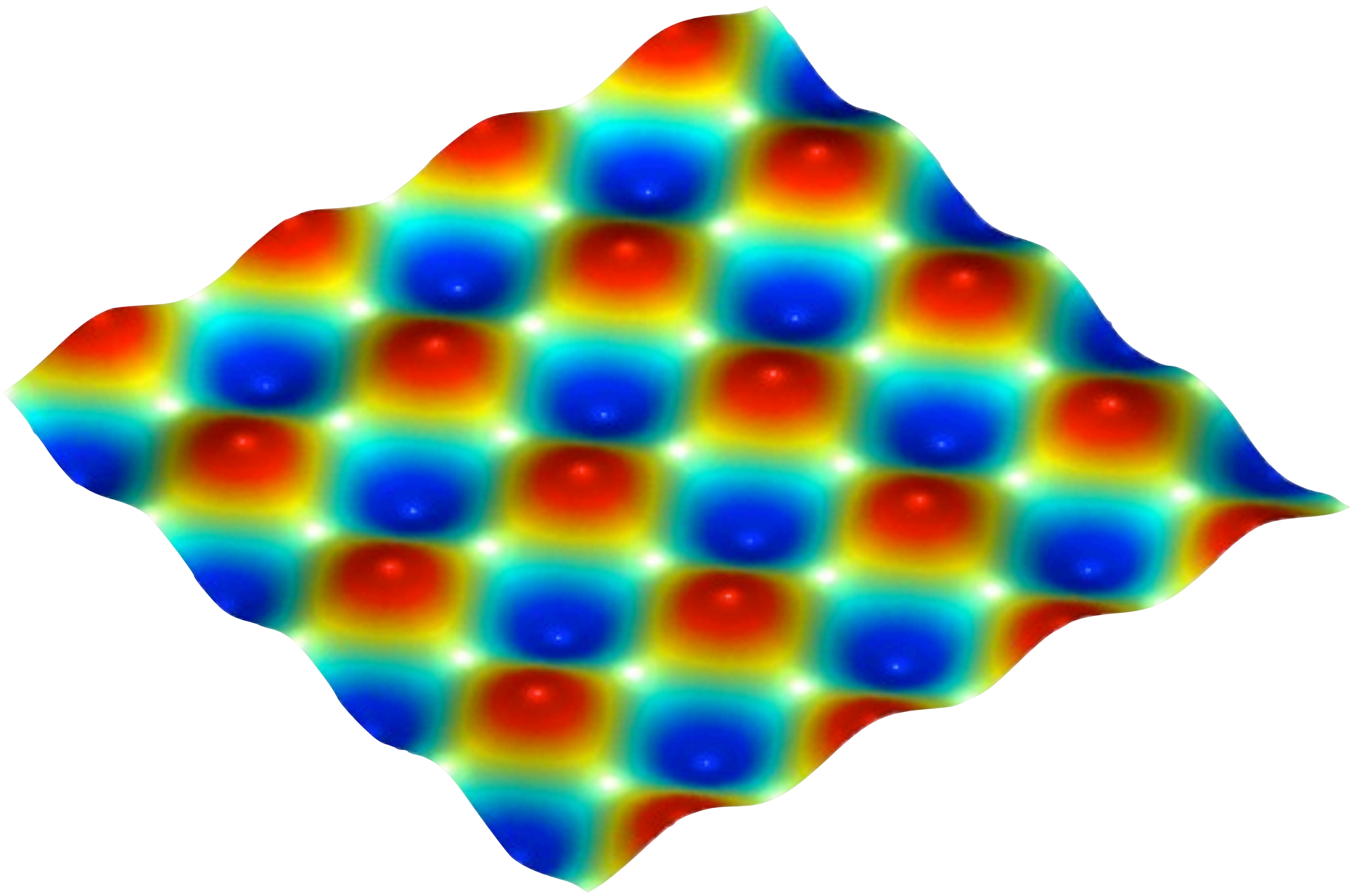
optically generated synthetic Lorentz force along x ~ v_y

magnetically generated force along y ~ v_x

avoids the centrifugal effect present in the rotated systems



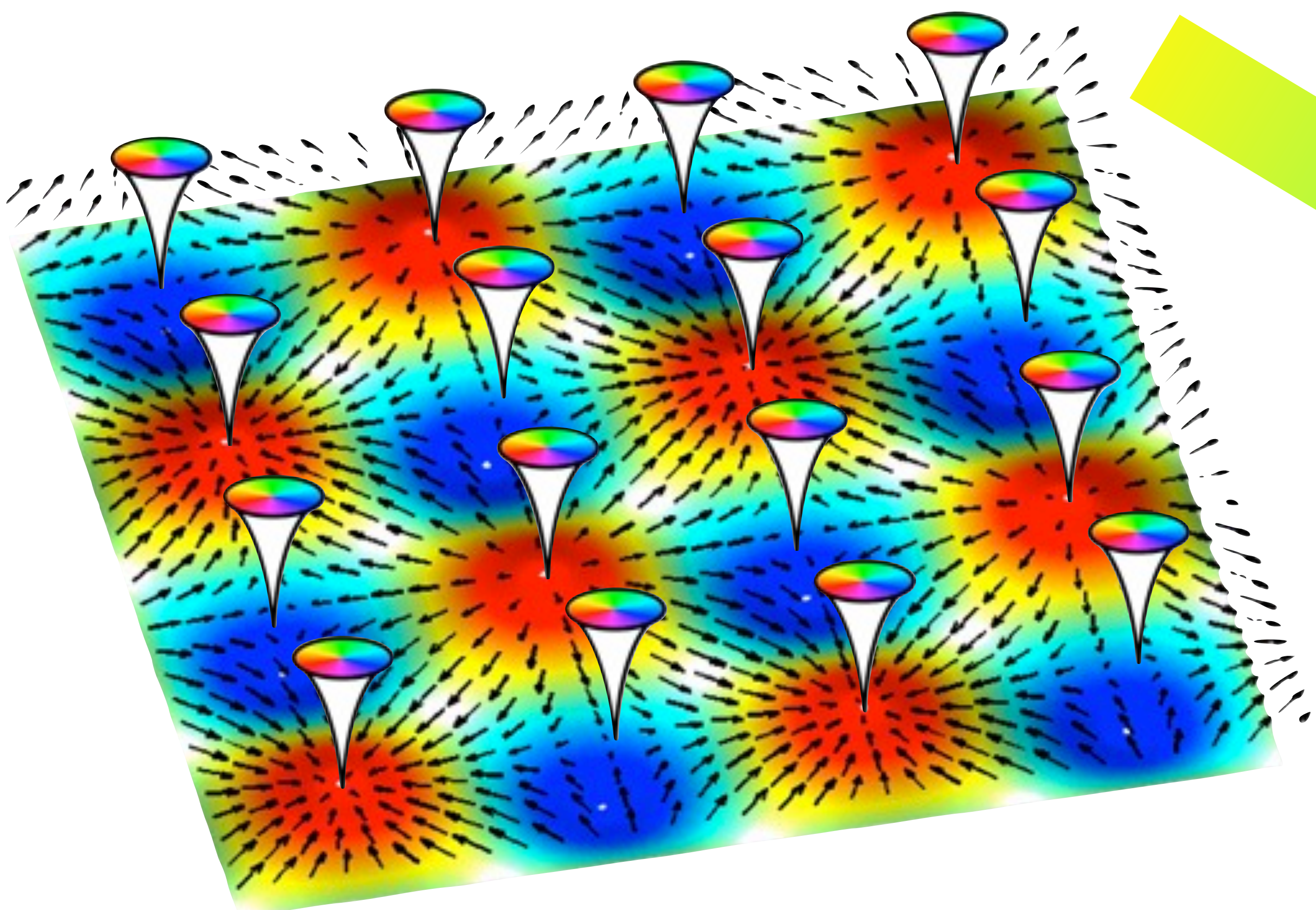
topologically trivial (simply connected) optical lattice



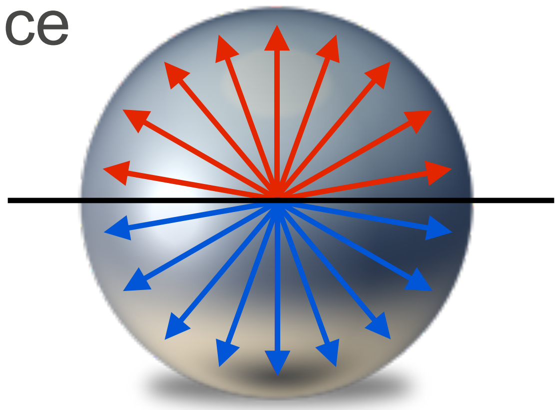
$$V_{\rm ol} = V_0(\cos^2(kx)\mathbf{e}_x + \cos^2(ky)\mathbf{e}_y)$$

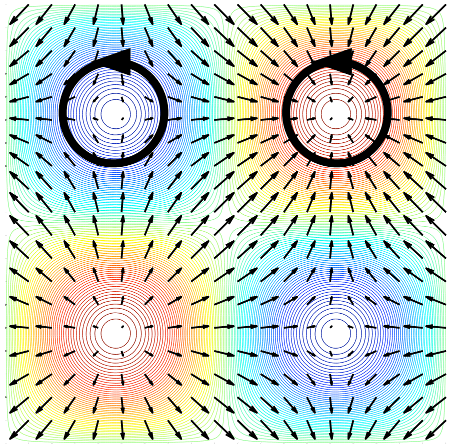
topologically nontrivial (multiply connected) optical flux lattice

Phys. Rev. Lett. 106, 175301 (2011)

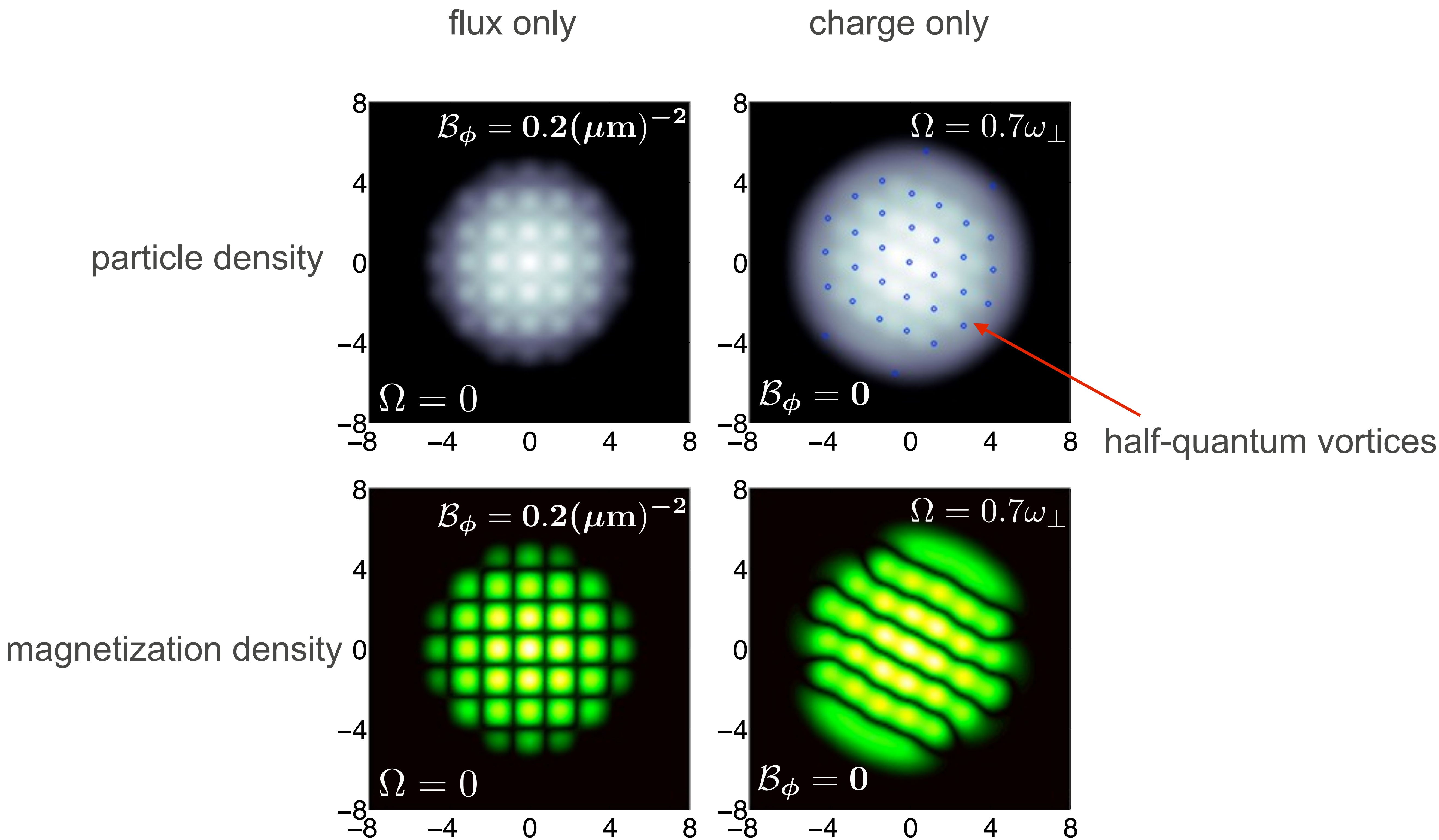


three-wave interference
=
vortex lattice

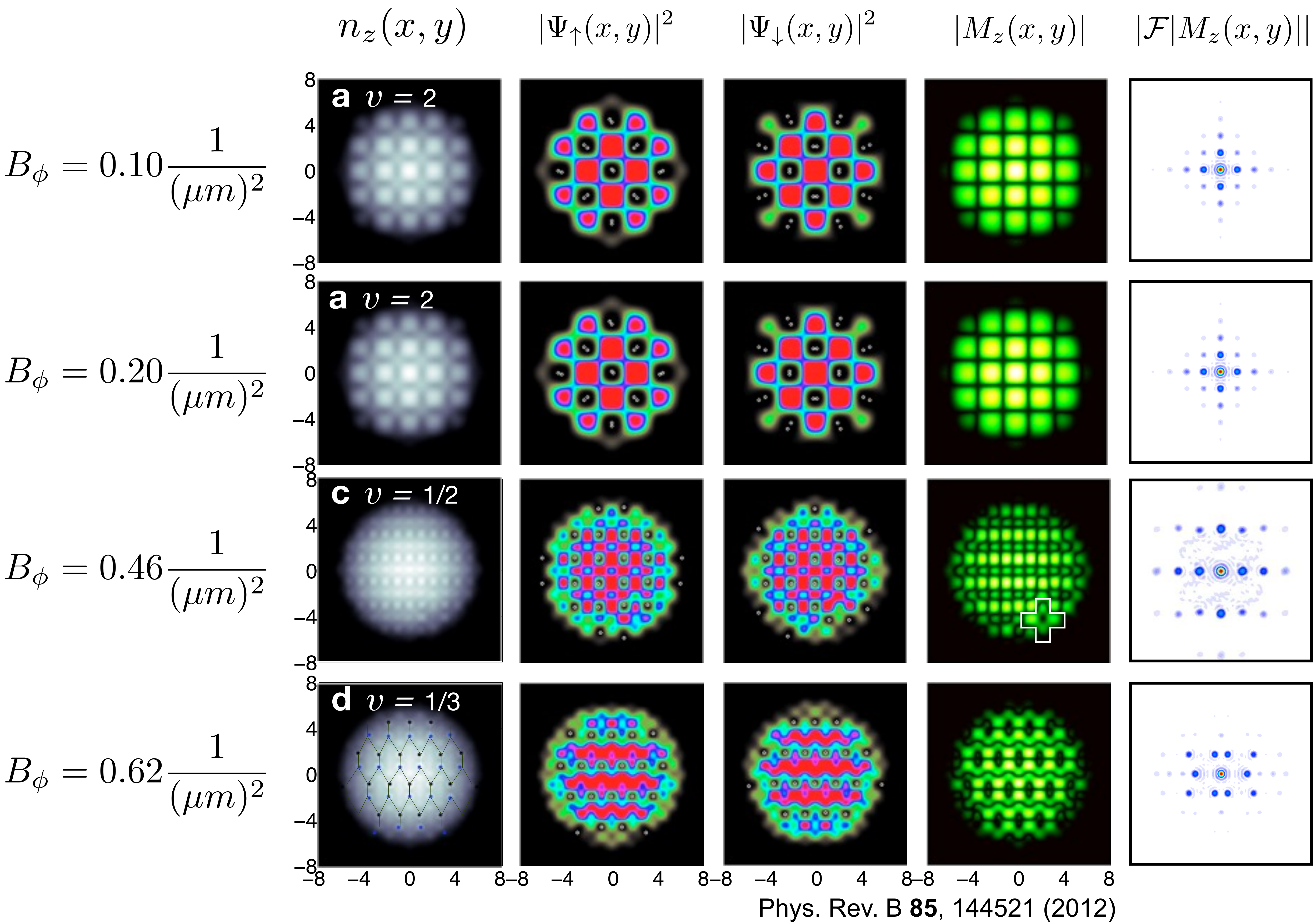


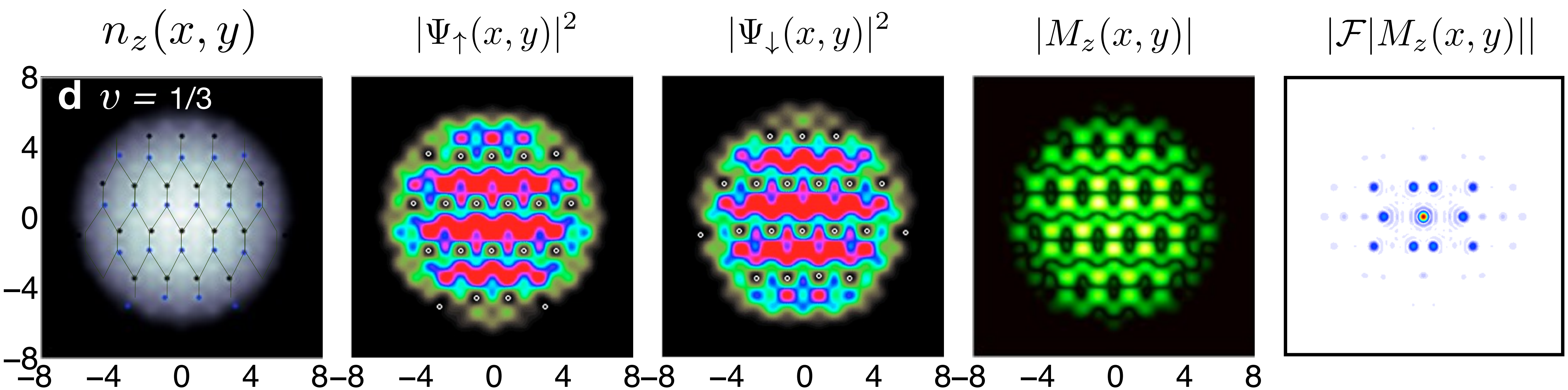


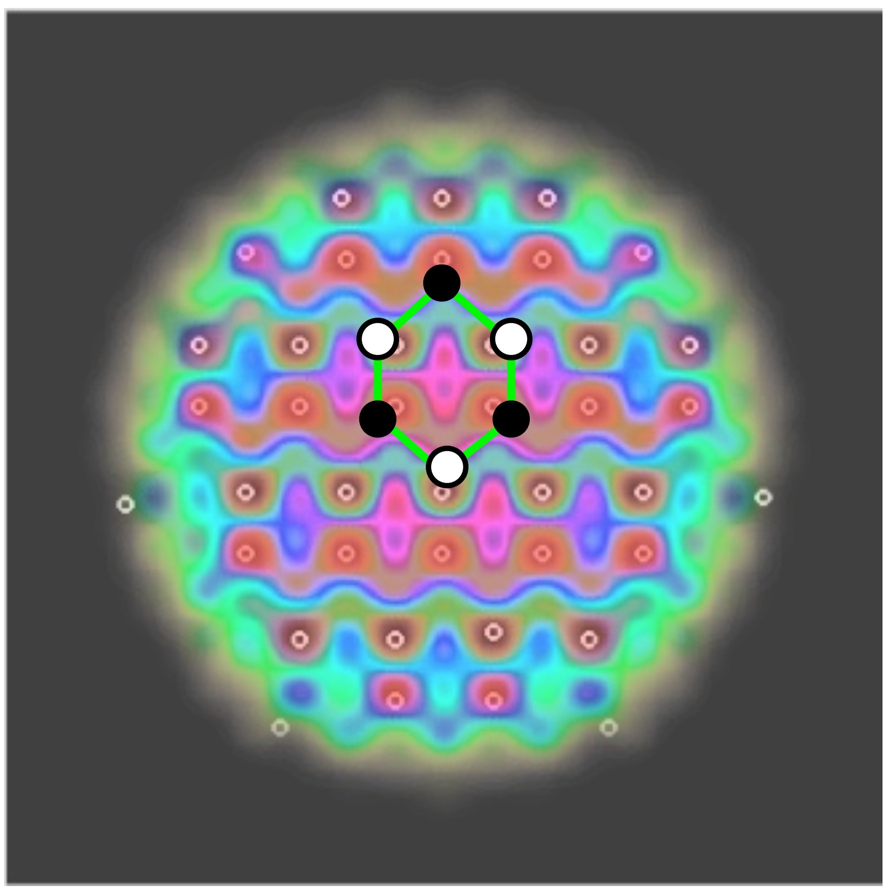
$$\mathbf{B}_\phi = B_0(\cos(kx)\mathbf{e}_x + \cos(ky)\mathbf{e}_y + \sin(kx)\sin(ky)\mathbf{e}_z)$$



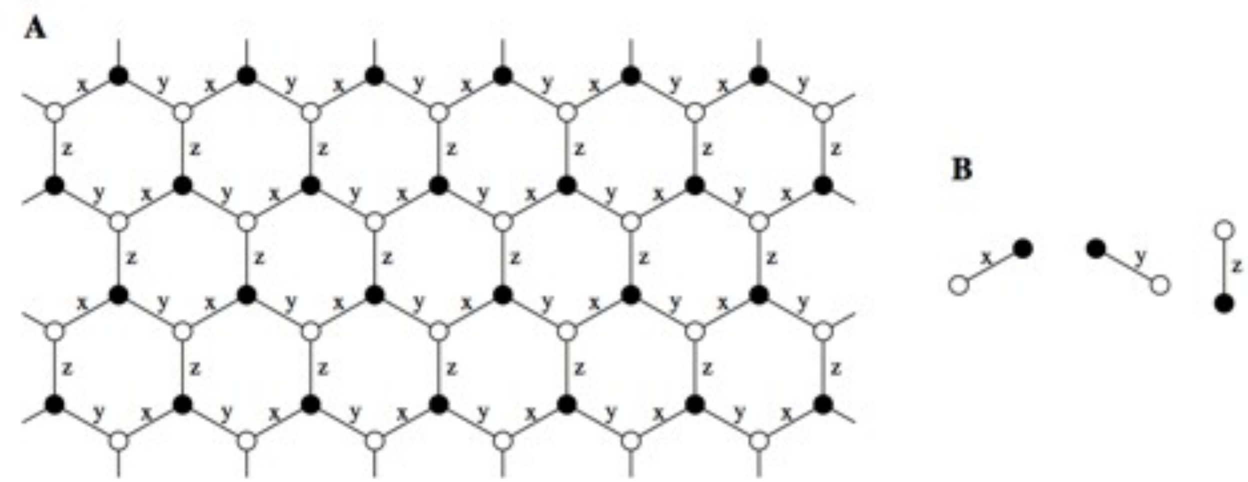
Phys. Rev. B **85**, 144521 (2012)


$n_z(x,y)$
$|\Psi_\uparrow(x,y)|^2$
$|\Psi_\downarrow(x,y)|^2$
$|M_z(x,y)|$
$|\mathcal{F}|M_z(x,y)||$
$B_\phi = 0.10 \frac{1}{(\mu m)^2}$
a $\upsilon = 2$
$B_\phi = 0.20 \frac{1}{(\mu m)^2}$
a $\upsilon = 2$
$B_\phi = 0.46 \frac{1}{(\mu m)^2}$
c $\upsilon = 1/2$
$B_\phi = 0.62 \frac{1}{(\mu m)^2}$
d $\upsilon = 1/3$
−8 −4 0 4 8
Phys. Rev. B **85**, 144521 (2012)





$$B_\phi = 0.62 \frac{1}{(\mu m)^2} \qquad \nu = 1/3$$



$$H = -J_x \sum_{x\text{-links}} \sigma_j^x \sigma_k^x - J_y \sum_{y\text{-links}} \sigma_j^y \sigma_k^y - J_z \sum_{z\text{-links}} \sigma_j^z \sigma_k^z,$$

Kitaev lattice

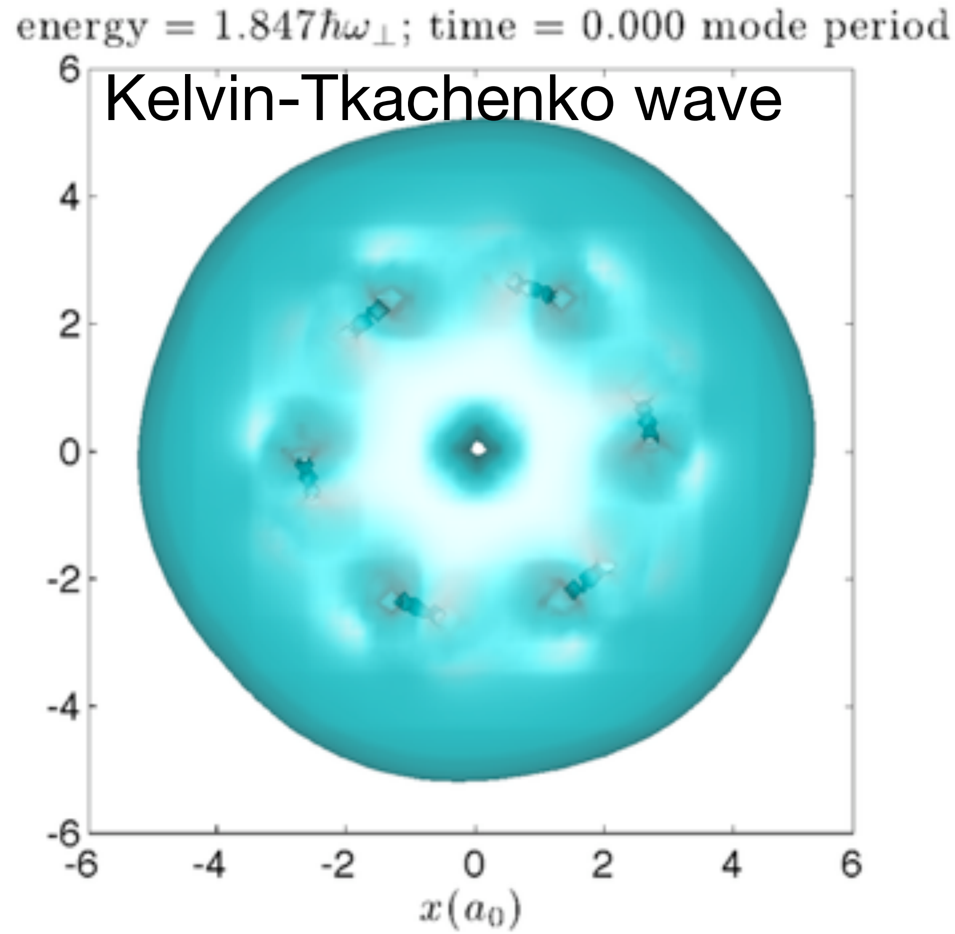
synthetic spin-orbit coupling

quantum turbulence

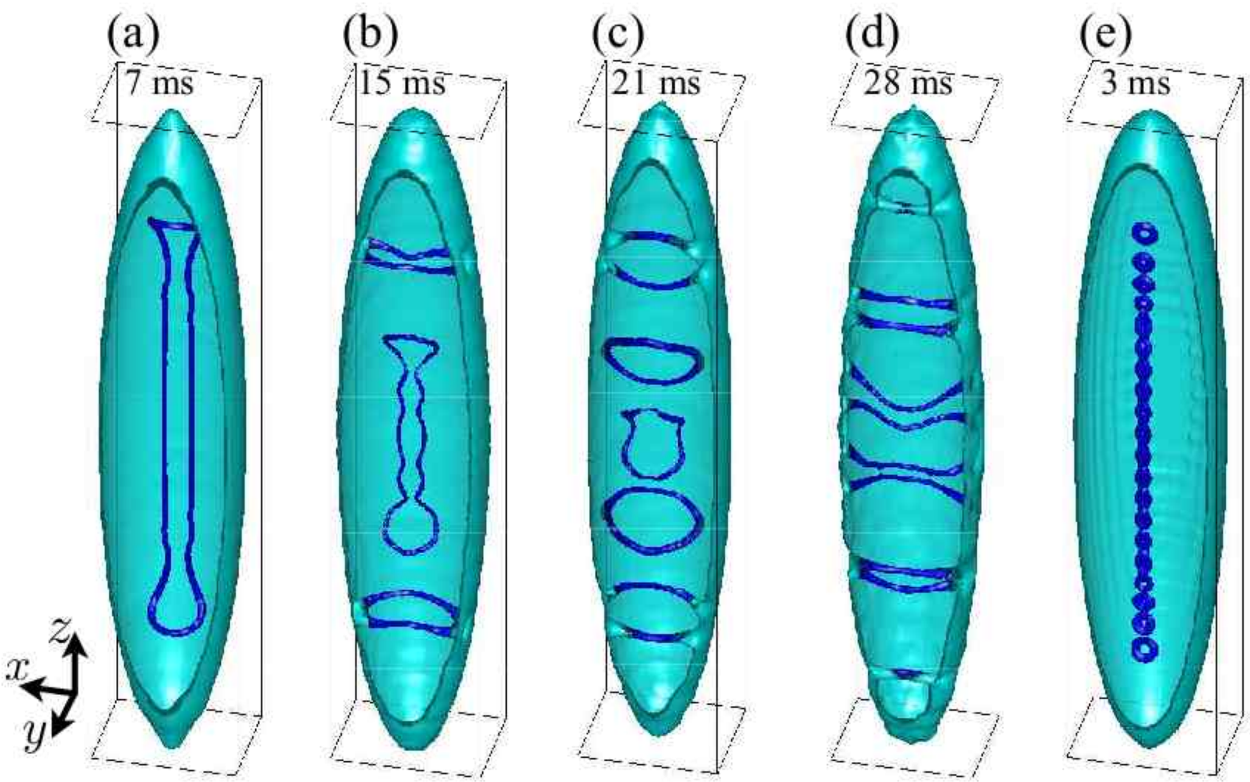
Phil. Trans. R. Soc. A 28 August 2008 vol. 366 no. 1877

http://youtu.be/R2KUAWj7l98

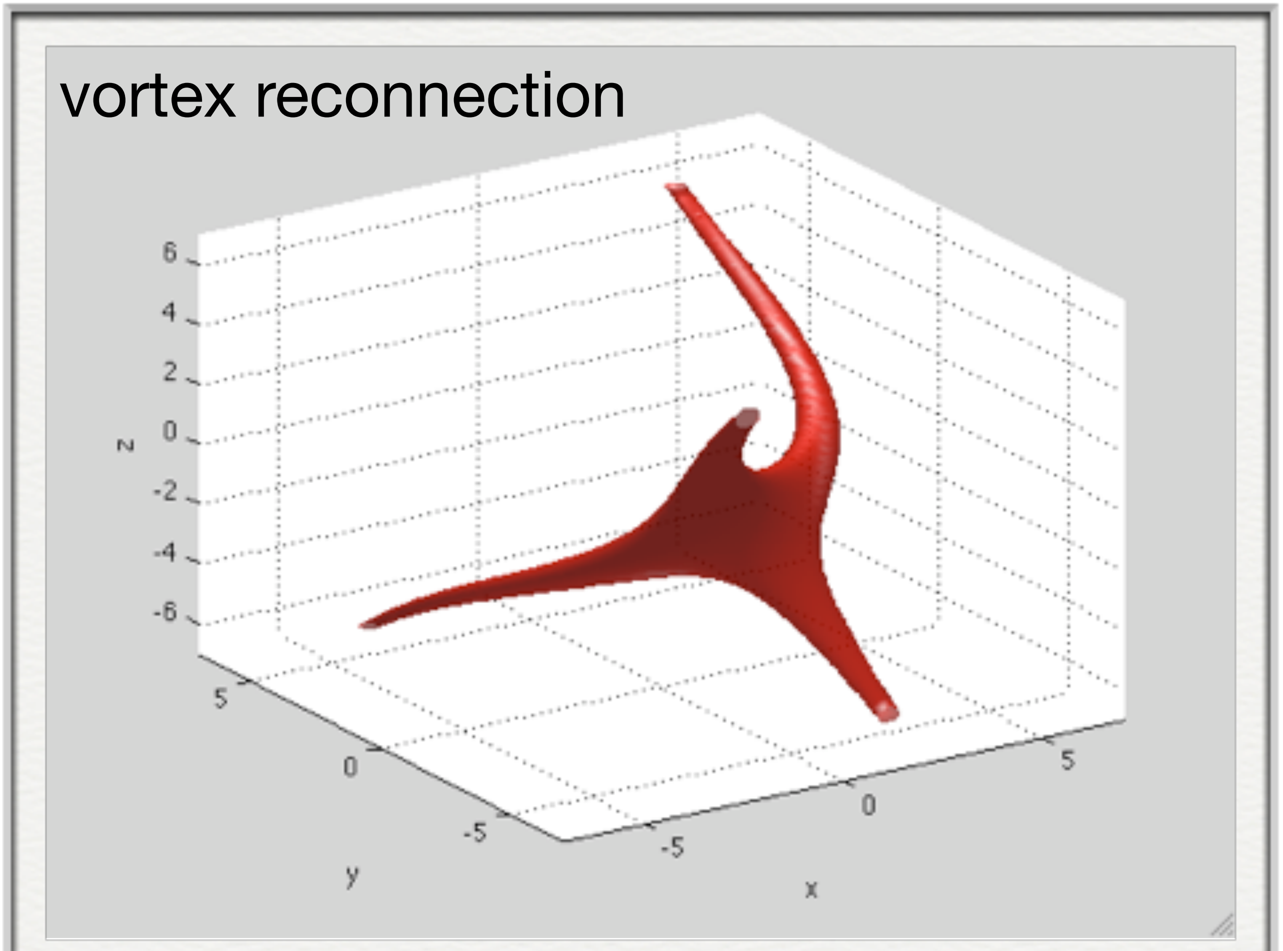
vortex waves, instabilities and reconnections



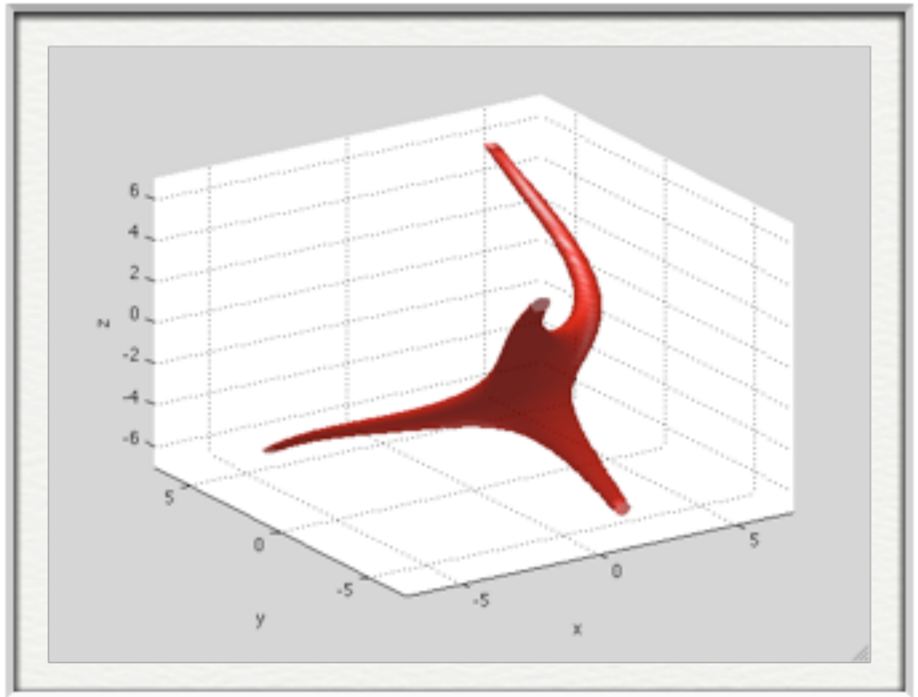
Crow instability



Phys. Rev. A **84**, 021603(R) (2011)

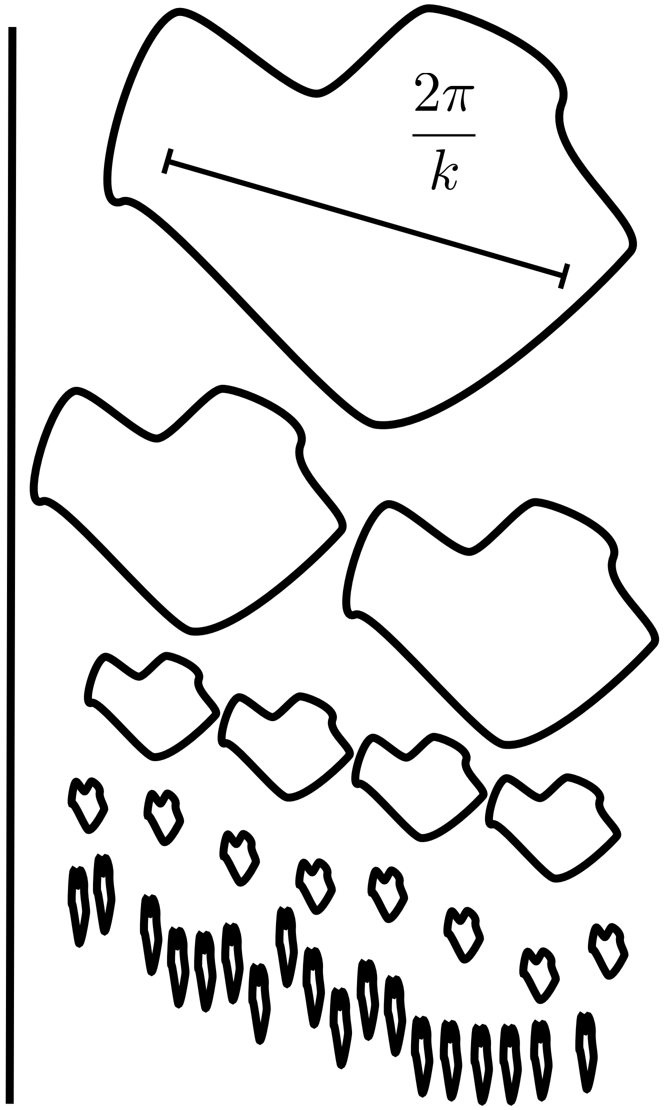


turbulence simplified: classical vs superfluid

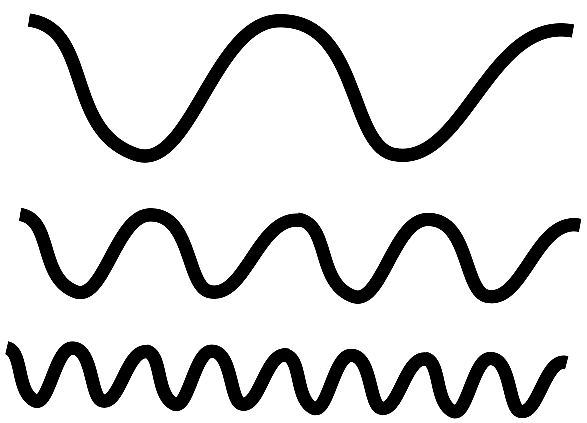


energy in: source of turbulence

inertial range: Richardson cascade and Kolmogorov -5/3 law



vortex reconnections



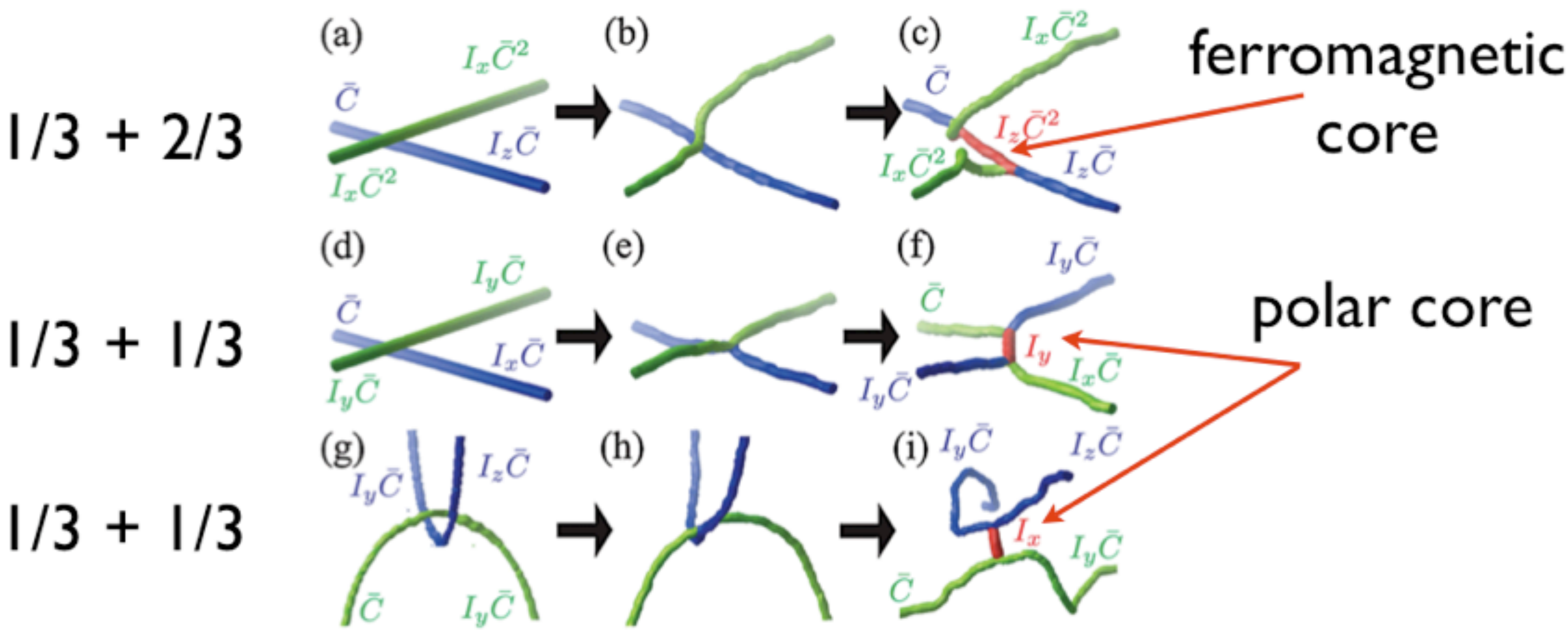
Kelvin wave cascade

energy out: dissipation / heat

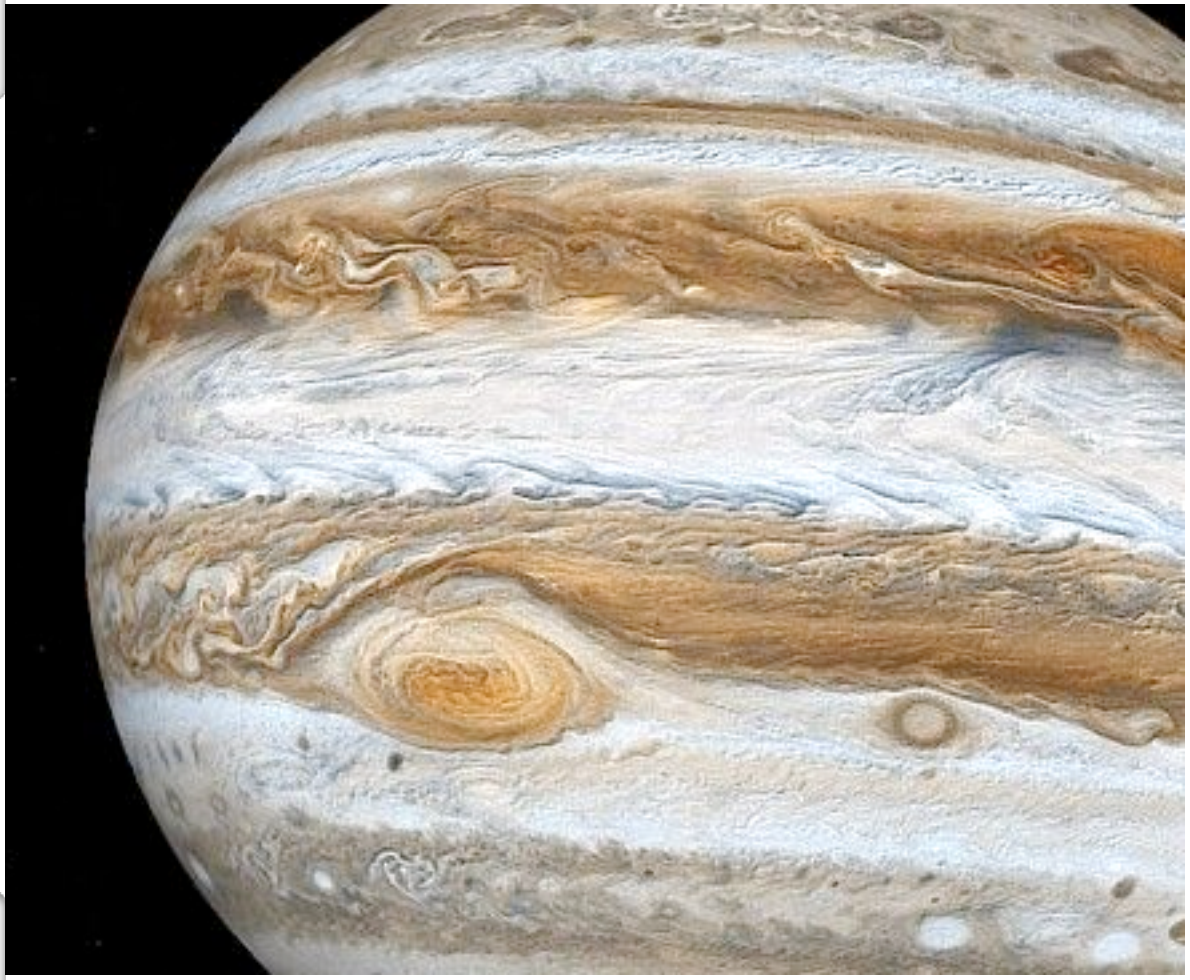
quantum turbulence with non-Abelian vortices

M. Kobayashi, Y. Kawaguchi, M. Nitta, and M. Ueda, PRL **103**, 115301 (2009)

networked superfluid turbulence



turbulence in 2D is special!



Rev. Mod. Phys. **78**, 87 (2006)

Rep. Prog. Phys. **43** 547 (1980)

inverse energy cascade

large coherent vortices

negative temperature states