Ultracold Atomic Fermi Gases

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Outline

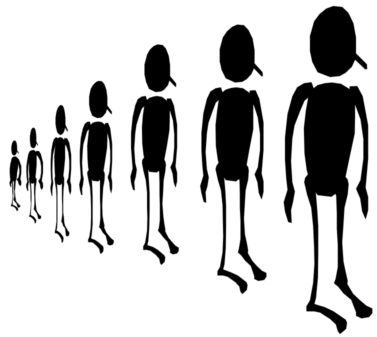
1. BCS Mean Field Theory
2. High Temperature Virial Expansion
3. Spin-Orbit Coupled Ultracold Fermi Gases I
4. Spin-Orbit Coupled Ultracold Fermi Gases II

1. Brief Review : Ultracold Fermi Gases
2. Energy and Length Scales
3. BCS Mean Field Theory
4. Solutions at BCS and BEC limit
5. Application: Calculate Collective Modes

There are two kinds of particles in the world: fermion and bosons

Fermions: half-integral spin electrons, protons, neutrons, 2H, 6Li,… are forbidden by the Pauli exclusion principle to have more than two of the same type in the same state. They are the "loners" of the quantum world. If electrons were not fermions, we would not have chemistry. Fermion obey the rules of Fermi-Dirac statistics.

Bosons: integral spin photons, 1H, 7Li, 23Na, 87Rb, 133Cs,… love to be in the same state. They are the joiners of the quantum world. If photons were not bosons, we would not have lasers. Bosons obey the rules of Bose-Einstein statistics.

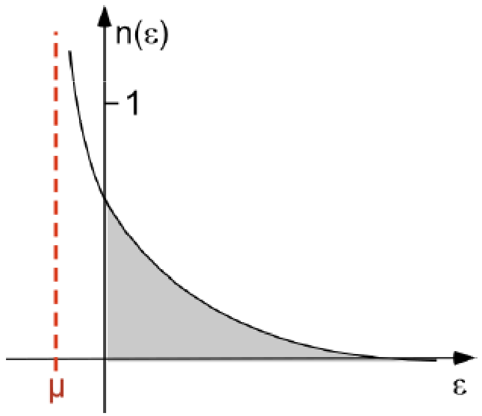


Quantum statistics

Quantum statistics

Bose-Einstein distribution

$$n(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} - 1}$$



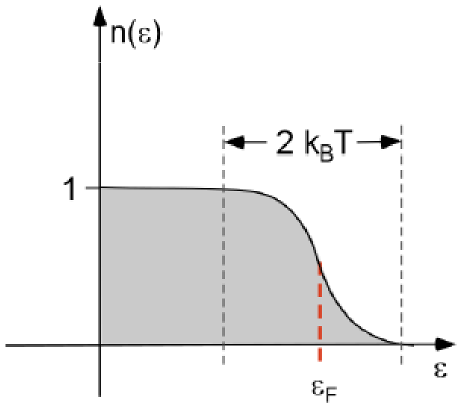
The chemical potential μ follows from the norm of n(ε) and depends on ***T*** and ***N***.

For T→0: $\mu \to \varepsilon_0$ (ground state energy)

macroscopic population of the ground state

Fermi-Dirac distribution

$$n(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1} \qquad \beta = \frac{1}{k_B T}$$



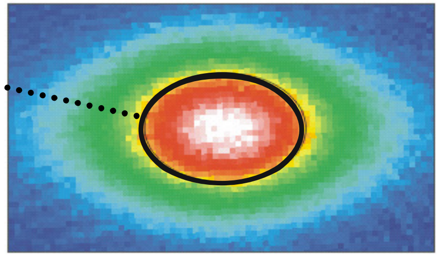
For T→0: $\mu \to \varepsilon_F$ (fermi energy)

$$n(\varepsilon) \to \Theta(\varepsilon - \mu) = \begin{cases} 1 & for \ \ \varepsilon < \mu \\ 0 & for \ \ \varepsilon > \mu \end{cases}$$

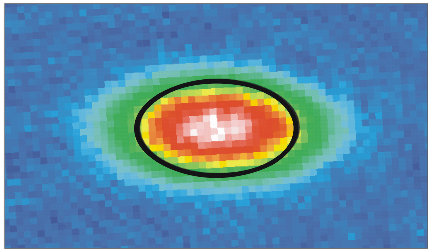
Quantum degeneracy

Velocity distributions

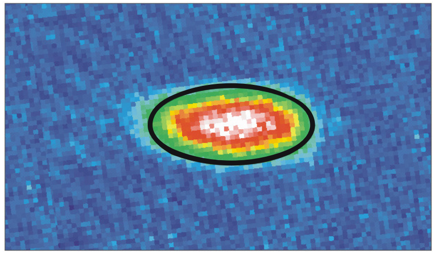
E_F



$T/T_F=0.77$

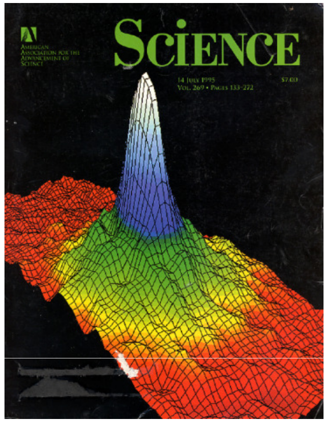


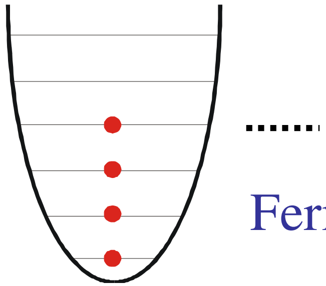
$T/T_F=0.27$



$T/T_F=0.11$

Quantum Degeneracy
in Trapped Fermi Gases in 1999

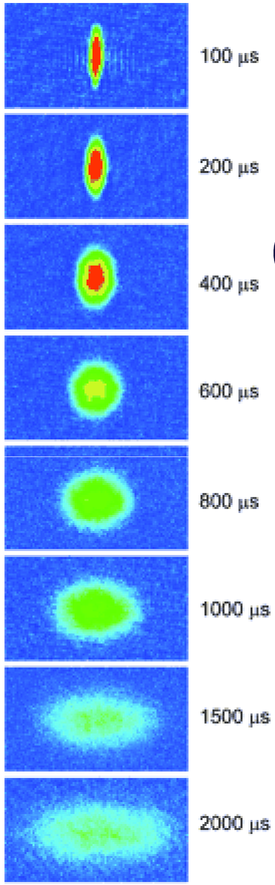




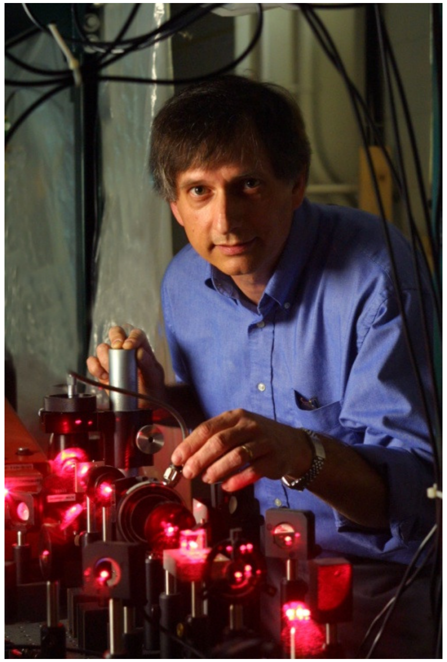
E_F

Fermi sea of atoms

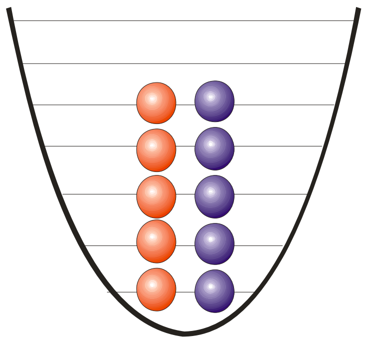
Superfluidity: Fermions



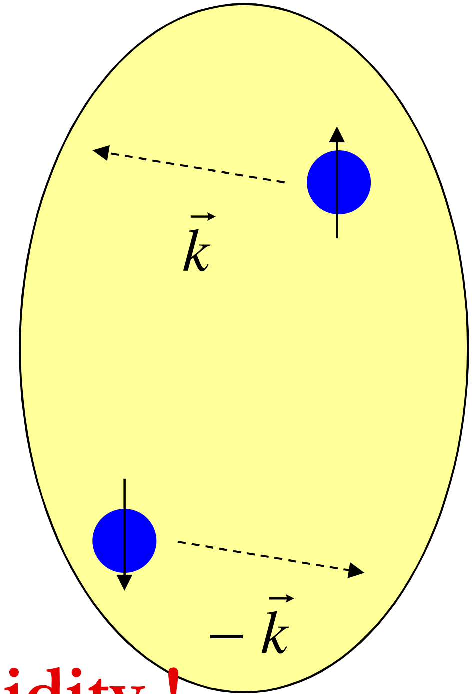
Superfluidity: Fermions in 2002



Interaction

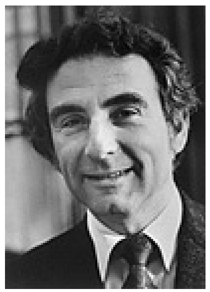
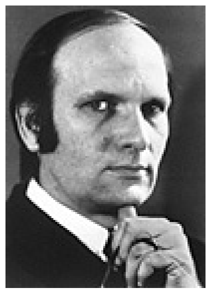


Cooper pairs - BCS superfluidity



Bardeen-Cooper-Schieffer Superfluidity !

2012 is 55th anniversary of BCS Theory

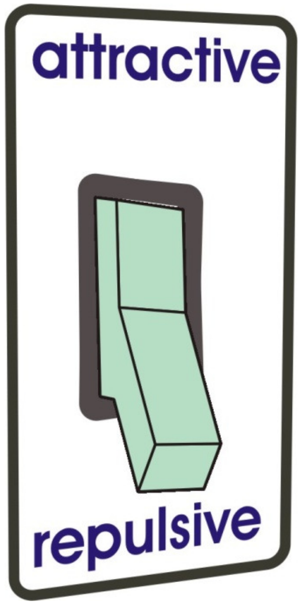
Interaction

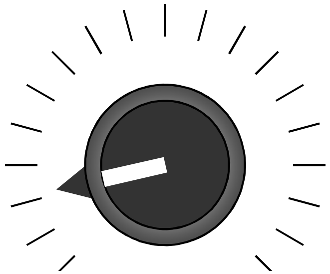
Interactions are characterized by the s-wave scattering length,

$a > 0$ repulsive, $a < 0$ attractive

Large $|a| \rightarrow$ strong interactions

In an ultracold atomic gas, we can control a





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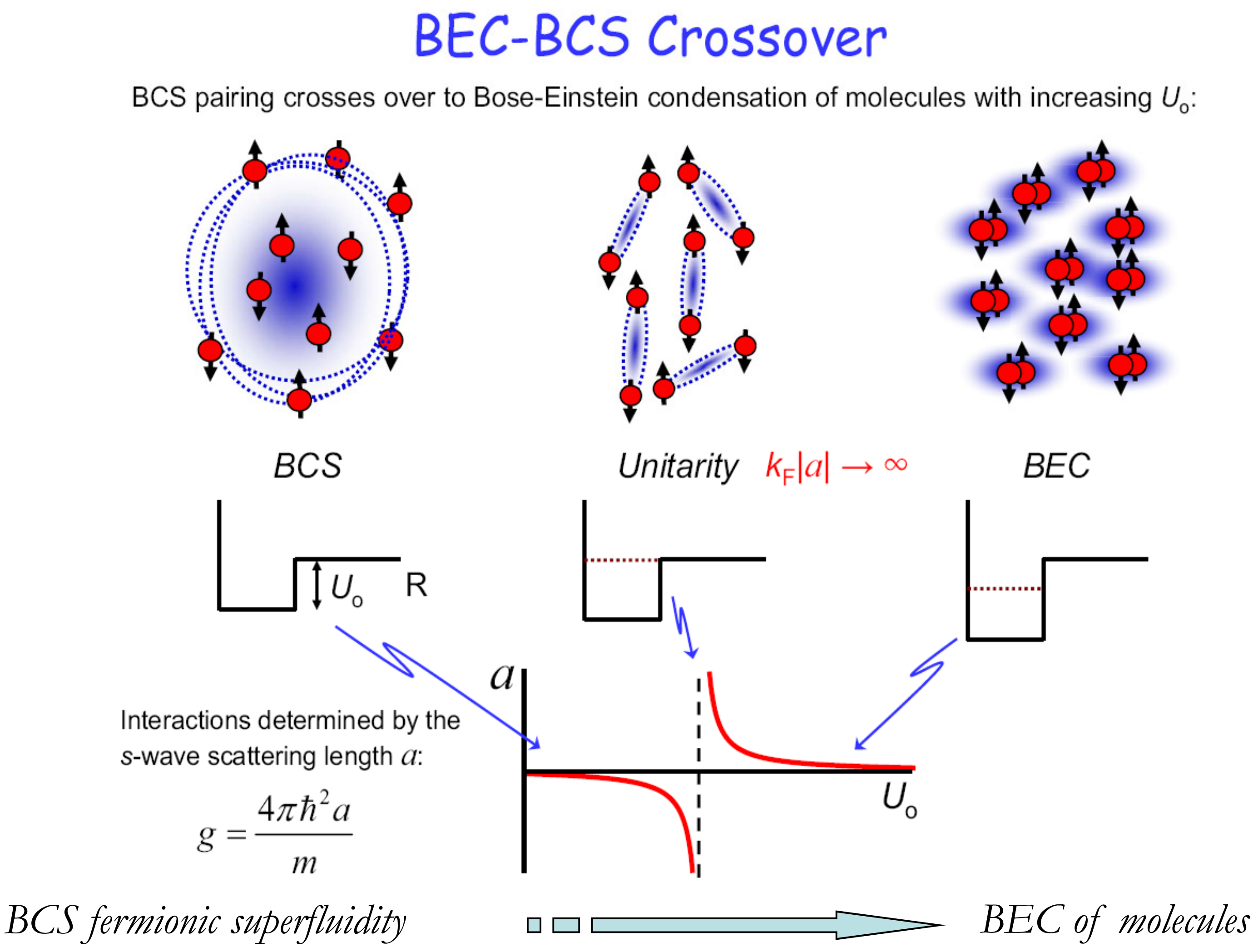
BCS-BEC Crossover

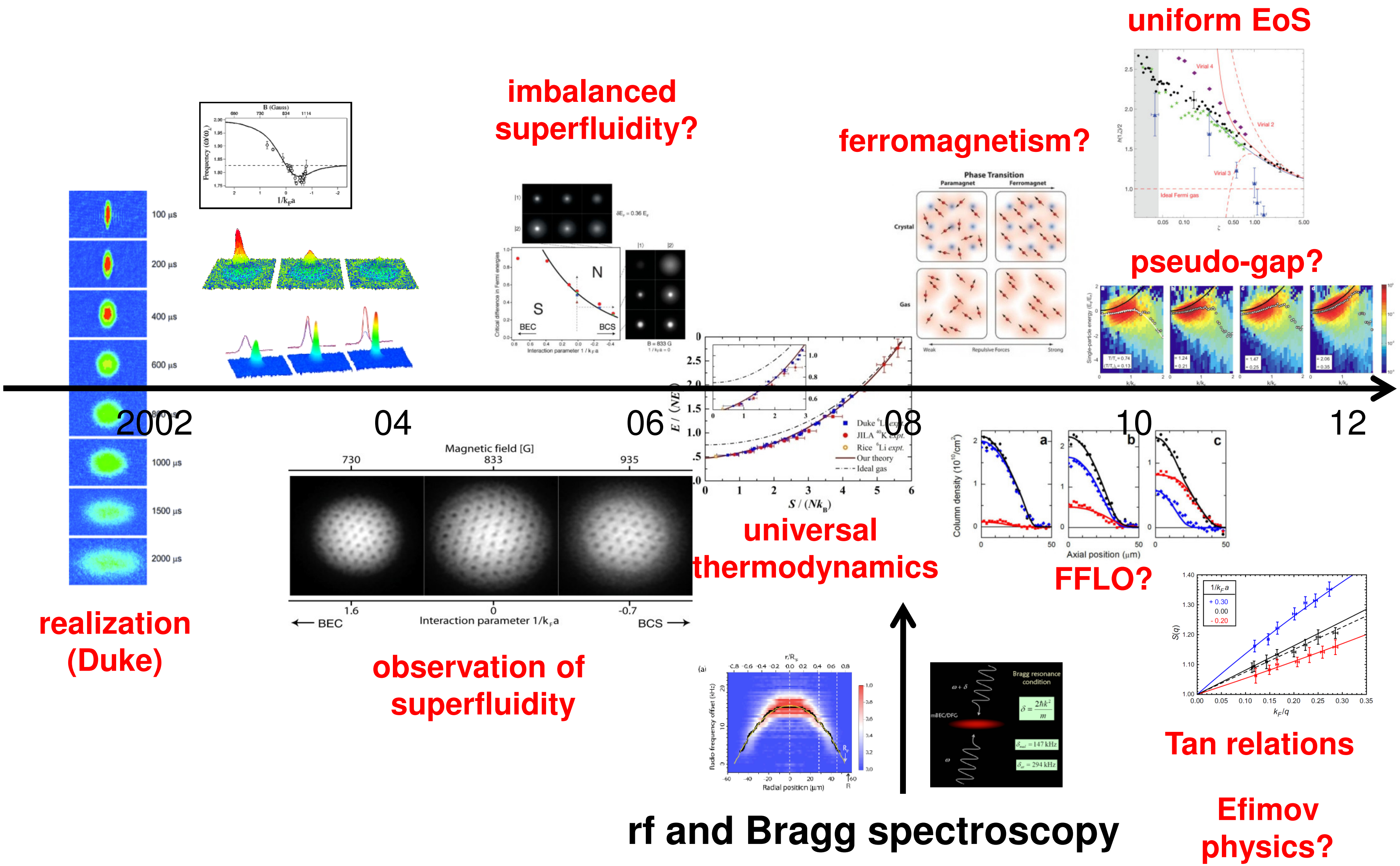


Interaction strength tunable via Feshbach resonances

Experimental works

Group	Units	atoms	Observations
Thomas	**Duke**	^{6}Li	*collective modes; heat capacity;*
Grimm	**Innsbruck**	^{6}Li	*collective modes; pairing gap;*
Salomon	**ENS**	^{6}Li	*release energy; equation of state*
Hulet	**Rice**	^{6}Li	*observation of molecules; FFLO state*
Ketterle	**MIT**	^{6}Li	*condensation of pairs;* **vortex***;*
Jin	**JILA**	^{40}K	*condensation of pairs; momentum distribution;*
Ueda	**Tokyo**	^{6}Li	*condensate Fraction; critical temperature*
Turlapov	**IAP**	^{6}Li	*2D Fermi gas*
Zwierlein	**MIT**	^{6}Li	*Fermi Polarons; Spin Transport*
Köhl	**Cambridge**	^{40}K	*rf-spectroscopy in 2D Fermi gas*
Zhang	**Shangxi**	^{40}K	*SO coupled Fermi gases*
Hannaford/Vale	**SUT**	^{6}Li	*p-wave; Bragg spectroscopy, Tan relation*

Global progress (experiment)



Global progress (theory)

1D exact solutions

Mean field

Large-*N*, ε-expansion, RG?

***T*-matrix approximation?** **Tan relations!** **Operator product expansion?**

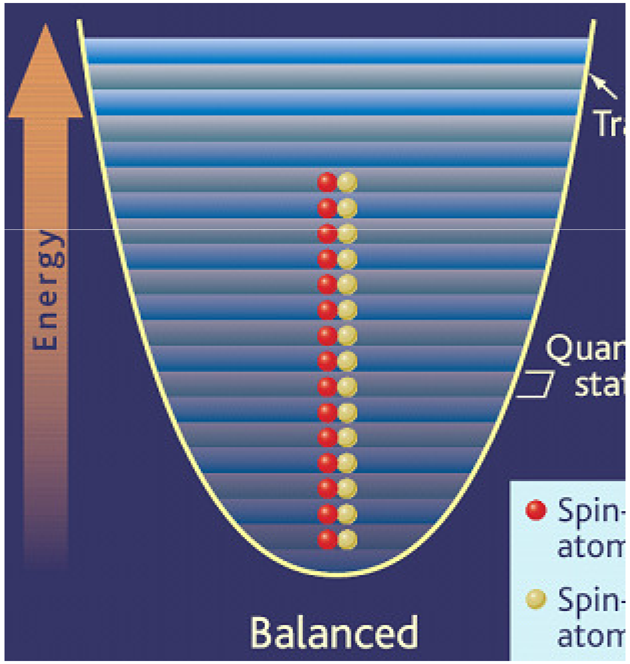
2002 04 06 08 10 12

Quantum Monte Carlo? **Virial expansion**

Few-body solutions

Color: Black (tried, experienced), blue (to be tried), red (interested)

Energy and Length Scales



Pauli Exclusion Principle

Fermi Energy $$E_F = (6N)^{1/3}\hbar\varpi$$

$$\varpi = (\omega_x\omega_y\omega_z)^{1/3}$$

Fermi Temperature $$T_F = E_F / k_B$$

Characteristic Size $$R_F = (2E_F / M\varpi^2)^{1/2}$$

Characteristic Wave Number $$k_F = (2mE_F / \hbar^2)^{1/2}$$

Harmonic Trap: Fermi Energy

N Single Component Fermions in a 3D Harmonic Trap

Energy Level $E = \left(2n_r + |l| + 3/2\right)\hbar\omega$ $\qquad E_F = A\hbar\omega$

Degeneracy $2l + 1$

The Number of Atoms

$$N = \sum_{n_r=0}^{A/2} \sum_{l=0}^{A-2n_r} (2l+1)$$

$$= \sum_{n_r=0}^{A/2} \left(1+3+5+\cdots 2\left(A-2n_r\right)+1\right)$$

Let's count

$$= \sum_{n_r=0}^{A/2} \left(\left(A-2n_r\right)+1\right)^2$$

$$= 1 + 3^2 + 5^2 + \cdots \left(A+1\right)^2$$

$$\cong \frac{A^3}{6}$$

$$E_F = (6N)^{1/3}\hbar\omega$$

Density of State

The number of states per interval of energy

$\Delta\varepsilon$

Δn

$$g(\varepsilon) = \frac{dn}{d\varepsilon}$$

$$\varepsilon = A\hbar\omega \qquad n \cong \frac{A^3}{6}$$

$$g(\varepsilon) = \frac{\varepsilon^2}{2(\hbar\omega)^3}$$

2D and 1D

N Single Component Fermions in a 2D Harmonic Trap

Energy Level $E = (2n + |m| + 1)\hbar\omega$ $E_F = A\hbar\omega$

The Number of Atoms
$$N = \sum_{n=0}^{A/2} 2(A - 2n)$$
$$= 4(1 + 2 + 3 + \cdots \frac{A}{2})$$
$$= (A/2 + 1)A$$
$$\cong \frac{A^2}{2}$$

$E_F = (2N)^{1/2}\hbar\omega$

N Single Component Fermions in a 1D Harmonic Trap

Energy Level $E = (n + 1/2)\hbar\omega$ $E_F = A\hbar\omega$

$$N = \sum_{n=0}^{A} 1 = A$$

$E_F = N\hbar\omega$

Two-Component System

3D $$E_F = (3N)^{1/3} \hbar\omega$$

2D $$E_F = (N)^{1/2} \hbar\omega$$

1D $$E_F = N/2\hbar\omega$$

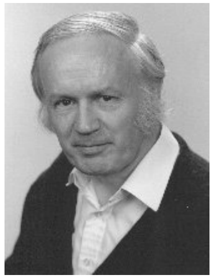
Homogeneous case: Fermi Energy

$$\int d^s p d^s x / h^s = N \quad \begin{cases} s=3 & k_F = \left(6n\pi^2\right)^{1/3} \\ s=2 & k_F = \left(4n\pi\right)^{1/2} \\ s=1 & k_F = n\pi \end{cases}$$

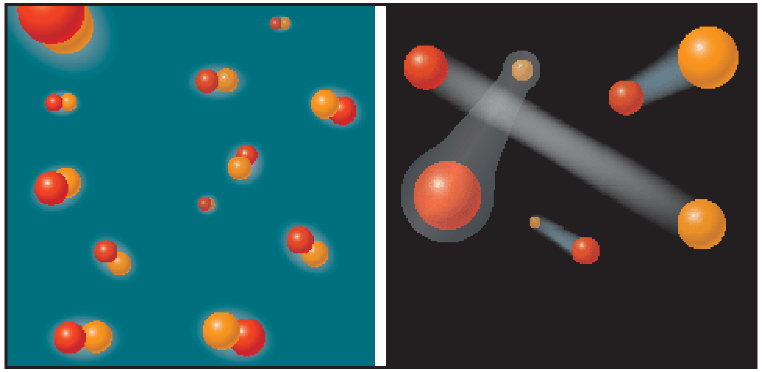
Two-Component System

$$\begin{cases} s=3 & k_F = \left(3n\pi^2\right)^{1/3} \\ s=2 & k_F = \left(2n\pi\right)^{1/2} \\ s=1 & k_F = n\pi/2 \end{cases}$$

Theoretical History of Crossover: MF



- **Eagles**, **Leggett** noted that BCS T=0 wavefunction could be generalized to arbitrary attraction: a *smooth* BCS-BEC crossover !



BEC

$$\Psi_0 = \exp\left(N_B^{1/2} \sum_{\mathbf{k}} \phi_{\mathbf{k}} c_{\mathbf{k}}^{+} c_{-\mathbf{k}}^{+}\right) |0\rangle$$

$$v_k = \frac{N_B^{1/2}\phi_k}{\left(1 + N_B \phi_k^2\right)^{1/2}}$$

BCS

$$\Psi_0 = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c_{\mathbf{k}}^{\dagger} c_{-\mathbf{k}}^{\dagger})|0\rangle$$

- **Holland, Drummond** *et al.* applied it to cold atom gases, with a molecular field.
- **Levin** *et al.* developed a MF theory, including bosonic degree of freedoms.

Wavefunction at Zero Temperature

BCS

$$\Psi_0 = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}}) |0\rangle$$

$$\Psi_0 = 1 + \sum_k v_k c^{+}_{k\uparrow} c^{+}_{-k\downarrow} + \frac{1}{2}\left(\sum_k v_k c^{+}_{k\uparrow} c^{+}_{-k\downarrow}\right)^2 + \cdots + \frac{1}{n!}\left(\sum_k v_k c^{+}_{k\uparrow} c^{+}_{-k\downarrow}\right)^n$$

Pauli Exclusion Principle

$$c^{2}_{k,\sigma} = 0$$

$$c^{+2}_{k,\sigma} = 0$$

$$\Psi_0 = \exp\left[\sum_k v_k c^{+}_{k\uparrow} c^{+}_{-k\downarrow}\right] \qquad v_k = N_B^{1/2} \phi_k$$

$$|\Psi_0 >= \exp\left(N_B^{1/2} \sum_{\mathbf{k}} \phi_{\mathbf{k}} c^{+}_{\mathbf{k}\uparrow} c^{+}_{-\mathbf{k}\downarrow}\right) |0\rangle \qquad b_0^{+} = \sum_{\mathbf{k}} \phi_{\mathbf{k}} c^{+}_{\mathbf{k}\uparrow} c^{+}_{-\mathbf{k}\downarrow}$$

$$|\Psi_0 >= \exp\left(N_B^{1/2} b_0^{+}\right) |0\rangle = |\alpha > \qquad \alpha = N_B^{1/2}$$

BEC

$$\Psi_0 = \exp\left(N_B^{1/2} \sum_{\mathbf{k}} \phi_{\mathbf{k}} c^{+}_{\mathbf{k}} c^{+}_{-\mathbf{k}}\right) |0\rangle$$

BCS-BEC crossover

(Leggett 1980): Surprisingly, at zero temperature both BEC and BCS can be described by a same class of wave function (i.e., the BCS wave function):

a *smooth* BCS-BEC crossover!

Hamiltonian

$$H = \sum_{k\sigma} (\varepsilon_K - \mu) c^{+}_{k\sigma} c_{k\sigma} + g \sum_{k_1, k_2, k_3, k_4} c^{+}_{k_1\uparrow} c^{+}_{-k_2\downarrow} c_{-k_3\downarrow} c_{k_4\uparrow}$$

inter-atomic interaction

μ *chemical potentials*

$$\epsilon_{\mathbf{k}} = \hbar^2 \mathbf{k}^2 / 2m$$

$$k_1 - k_2 = k_3 - k_4$$

Mean Field Method

BCS

$$\Psi_0 = \prod_{\mathbf{k}} (u_{\mathbf{k}} + v_{\mathbf{k}} c^{\dagger}_{\mathbf{k}} c^{\dagger}_{-\mathbf{k}})|0\rangle$$

Pauli Exclusion Principle

$$c^2_{k,\sigma} = 0$$

$$c^{+2}_{k,\sigma} = 0$$

assume $\chi_k = 1 + v_k c^+_{k\uparrow} c^+_{-k\downarrow}$

$$< \Psi_0 | \Psi_0 > = < 0 | \prod_{k_1} \chi_{k_1} \prod_{k_2} \chi_{k_2} | 0 >$$

$$= < 0 | \prod_k \left[1 + |v_k|^2 c_{-k\downarrow} c_{k\uparrow} c^+_{k\uparrow} c^+_{-k\downarrow} \right] | 0 > = \prod_k \left[1 + |v_k|^2 \right]$$

$$< \Psi_0 | c^+_{k\sigma} c_{k\sigma} | \Psi_0 > = < 0 | \prod_{k_1} \chi_{k_1} c^+_{k\sigma} c_{k\sigma} \prod_{k_2} \chi_{k_2} | 0 > = \sum_k |v_k|^2 \prod_{k_1 \neq k} \left[1 + |v_k|^2 \right]$$

$$\chi_k c^+_{k\sigma} c_{k\sigma} \chi_k = c^+_{k\sigma} c_{k\sigma} + v_k c^+_{k\sigma} c_{k\sigma} c^+_{k\uparrow} c^+_{-k\downarrow} + v_k c_{-k\downarrow} c_{k\uparrow} c^+_{k\sigma} c_{k\sigma} + |v_k|^2 c_{-k\downarrow} c_{k\uparrow} c^+_{k\sigma} c_{k\sigma} c^+_{k\uparrow} c^+_{-k\downarrow}$$

$$< \Psi_0 | c^+_{k_1\uparrow} c^+_{-k_2\downarrow} c_{-k_3\downarrow} c_{k_4\uparrow} | \Psi_0 > = v_{k1} v_{k4} \left[\delta_{k_1 k_2} + \delta_{k_1 k_4} |v_{k_2}|^2 \right] \prod_{k \neq k_1, k_4} \left[1 + |v_k|^2 \right]$$

Mean Field Method

The Ground State Energy

$$\langle H \rangle_{GS} = \frac{\langle \Psi_0 | H | \Psi_0 \rangle}{\langle \Psi_0 | \Psi_0 \rangle} = \sum_k (\varepsilon_k - \mu) \frac{2\nu_k^2}{1+\nu_k^2} + g\left[\left(\sum_k \frac{\nu_k}{1+\nu_k^2} \right)^2 + \left(\sum_k \frac{\nu_k^2}{1+\nu_k^2} \right)^2 \right]$$

The Minimization of the ground state energy with respect to μ ν_k

$$\frac{\partial}{\partial \nu_k} \langle H \rangle_{GS} = 0$$

$$-\frac{\partial}{\partial \mu} \langle H \rangle_{GS} = \langle N \rangle$$

Mean Field Method

$$-\frac{\partial}{\partial\mu}\langle H\rangle_{GS} = \langle N\rangle$$

$$\langle N\rangle = n_\uparrow + n_\downarrow = 2\sum_k \frac{v_k^2}{1+v_k^2}$$

$$\frac{\partial}{\partial v_k}\langle H\rangle_{GS} = 0$$

$$2v_k\left(\varepsilon_k - \mu + g\sum_q \frac{v_q^2}{1+v_q^2}\right) = -g\left(1-v_k^2\right)\sum_q \frac{v_q}{1+v_q^2}$$

Introduce *The gap is order parameter*

$$\Delta = g\sum_k \langle c_{k\uparrow}c_{-k\downarrow}\rangle = g\sum_k \langle c_{-k}^{+}c_{k\uparrow}^{+}\rangle$$

Hartree term

$$U_k = \varepsilon_k - \mu + gn_\sigma$$

BCS quasiparticle energy

$$E_k = \sqrt{U_k^2 + \Delta^2}$$

$$v_k = \frac{E_k - U_k}{\Delta}$$

$$\frac{v_k^2}{1+v_k^2} = \frac{1}{2}\left(1 - \frac{U_k}{E_k}\right)$$

$$\frac{v_k}{1+v_k^2} = \frac{\Delta}{2E_k}$$

Mean Field Method

$$n = 2n_{\uparrow} = \sum_k \left(1 - \frac{U_k}{E_k}\right)$$

$$\frac{1}{g} = -\sum_k \frac{1}{2E_k}$$

inter-atomic interaction

The way in which the two body interaction *g* enters to characterize the scattering (in vacuum) is different from the way in which it enters to characterize the *N*-body processes leading to superfluidity.

$$\frac{m}{4\pi\hbar^2 a_s} \equiv \frac{1}{g} + \sum_k \frac{1}{2\varepsilon_k}$$

renormalization

Mean Field Method

Leggett model

$$
\begin{cases}
1 = \dfrac{4\pi\hbar^2 a}{m}\displaystyle\int \dfrac{d^3\mathbf{k}}{(2\pi)^3}\left[\dfrac{1}{2\varepsilon_{\mathbf{k}}} - \dfrac{1}{2E_{\mathbf{k}}}\right], & \text{(gap equation)} \\
n = \displaystyle\int \dfrac{d^3\mathbf{k}}{(2\pi)^3}\left(1 - \dfrac{\varepsilon_{\mathbf{k}} - \mu}{E_{\mathbf{k}}}\right). & \text{(number equation)}
\end{cases}
$$

note: $E_{\mathbf{k}} = \sqrt{(\varepsilon_{\mathbf{k}} - \mu)^2 + \Delta^2}$, *and the* Hartree *term is not included.*

Mean Field Method

$$H=\sum_{k\sigma}(\varepsilon_K-\mu)c^{+}_{k\sigma}c_{k\sigma}+g\sum_{k_1,k_2,k_3,k_4}c^{+}_{k_1\uparrow}c^{+}_{-k_2\downarrow}c_{-k_3\downarrow}c_{k_4\uparrow}$$

Wick's theorem

$$\sum_{k_1,k_2,k_3,k_4}c^{+}_{k_1\uparrow}c^{+}_{-k_2\downarrow}c_{-k_3\downarrow}c_{k_4\uparrow}=\sum_{k'k}\langle c^{+}_{k'\uparrow}c_{k'\uparrow}\rangle c^{+}_{k\downarrow}c_{k\downarrow}+\sum_{k'k}\langle c^{+}_{k'\downarrow}c_{k'\downarrow}\rangle c^{+}_{k\uparrow}c_{k\uparrow}-\sum_{k'k}\langle c^{+}_{k'\uparrow}c_{k'\uparrow}\rangle\langle c^{+}_{k\downarrow}c_{k\downarrow}\rangle$$
$$+\sum_{k'k}\langle c^{+}_{k'\uparrow}c^{+}_{-k'\downarrow}\rangle c_{-k\downarrow}c_{k\uparrow}+\sum_{k'k}\langle c_{-k'\downarrow}c_{k'\uparrow}\rangle c^{+}_{k\uparrow}c^{+}_{-k\downarrow}-\sum_{k'k}\langle c^{+}_{k'\uparrow}c^{+}_{-k'\downarrow}\rangle\langle c_{-k\downarrow}c_{k\uparrow}\rangle$$

Standard Bogoliubov Transformation

$$\begin{pmatrix}\alpha_{k\uparrow}\\ \alpha^{+}_{-k\downarrow}\end{pmatrix}=\begin{bmatrix}u_k & -\upsilon_k\\ \upsilon_k & u_k\end{bmatrix}\begin{pmatrix}c_{k\uparrow}\\ c^{+}_{-k\downarrow}\end{pmatrix}$$

$$\{\alpha_{k\sigma},\alpha^{+}_{k'\sigma'}\}=\delta_{kk'}\delta_{\sigma\sigma'}$$

for Boson

$$\Psi=\Phi+\tilde{\Psi}\ ;\ \tilde{\Psi}=\sum_i\left(u_i\alpha_i-\upsilon_i\alpha_i^{+}\right);\ \tilde{\Psi}^{+}=\sum_i\left(u_i\alpha_i^{+}-\upsilon_i\alpha_i\right)$$

Mean Field Method

We take

$$\cos\theta_k = u_k$$
$$\sin\theta_k = \upsilon_k$$

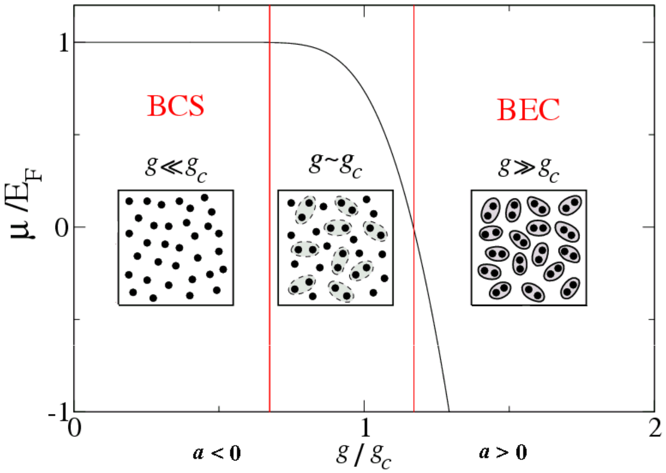
Off-Diagonal Term to Vanish

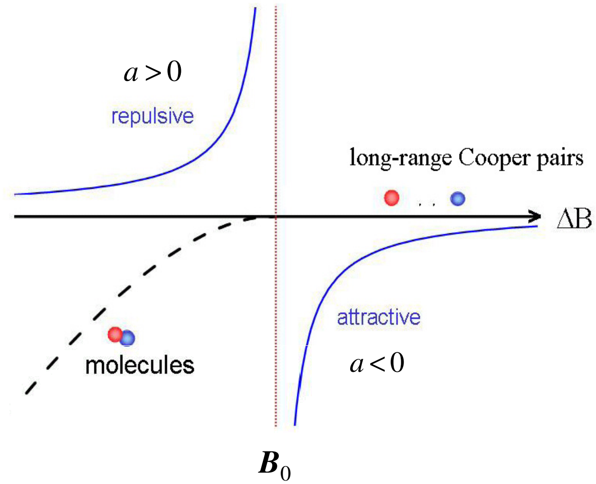
$$\tan 2\theta_k = \frac{\Delta}{U_K}$$

$$\begin{cases} n_\sigma = \sum_k \left\langle c_{k\sigma}^{+} c_{k\sigma} \right\rangle = \frac{1}{2} \sum_k \left(1 - \frac{U_k}{E_k} \right) \\ \frac{\Delta}{g} = \sum_k \left\langle a_{k\uparrow} a_{-k\downarrow} \right\rangle = -\sum_k \frac{\Delta}{2E_k} \end{cases}$$

$$\begin{cases} 1 = \frac{4\pi\hbar^2 a}{m} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left[\frac{1}{2\varepsilon_{\mathbf{k}}} - \frac{1}{2E_{\mathbf{k}}} \right], & \text{(gap equation)} \\ n = \int \frac{d^3\mathbf{k}}{(2\pi)^3} \left(1 - \frac{\varepsilon_{\mathbf{k}} - \mu}{E_{\mathbf{k}}} \right). & \text{(number equation)} \end{cases}$$

Mean Field Method





Mean Field Method

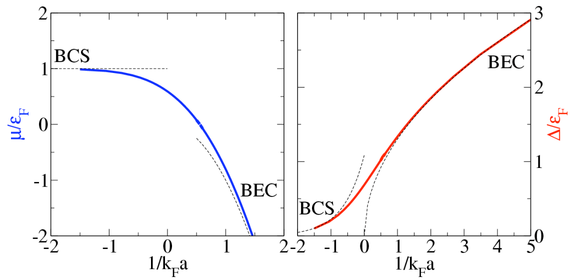
Leggett model

$$\begin{cases} 1 = \dfrac{4\pi\hbar^2 a}{m}\displaystyle\int \dfrac{d^3\mathbf{k}}{(2\pi)^3}\left[\dfrac{1}{2\varepsilon_{\mathbf{k}}} - \dfrac{1}{2E_{\mathbf{k}}}\right], & \textit{(gap equation)} \\ n = \displaystyle\int \dfrac{d^3\mathbf{k}}{(2\pi)^3}\left(1 - \dfrac{\varepsilon_{\mathbf{k}} - \mu}{E_{\mathbf{k}}}\right). & \textit{(number equation)} \end{cases}$$

note: $E_{\mathbf{k}} = \sqrt{(\varepsilon_{\mathbf{k}} - \mu)^2 + \Delta^2}$, *and the* Hartree *term is not included.*

Mean Field Method: BEC limit

BEC Limit $k_F a_s \to 0^+$



$$\varepsilon_F << \Delta << \mu$$

$$E_k = \sqrt{U_k^2 + \Delta^2}$$

$$E_k \approx \varepsilon_k - \mu + \frac{\Delta^2}{2(\varepsilon_k - \mu)}$$

$$\begin{cases} 1 = \dfrac{4\pi\hbar^2 a}{m}\displaystyle\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left[\frac{1}{2\varepsilon_{\mathbf{k}}} - \frac{1}{2E_{\mathbf{k}}}\right], \\ n = \displaystyle\int \frac{d^3\mathbf{k}}{(2\pi)^3}\left(1 - \frac{\varepsilon_{\mathbf{k}} - \mu}{E_{\mathbf{k}}}\right). \end{cases}$$

$$\Delta \cong \sqrt{16/(3\pi k_F a)}\,\varepsilon_F$$

$$\mu = -\frac{\hbar^2}{2ma_s^2} + \frac{a_s\pi\hbar^2}{m}n$$

$$\mu_B = 2\mu = -E_b + g_B n_B$$

$$E_b = \frac{\hbar^2}{ma_s^2} \quad \Longrightarrow \quad a_B = 2a_F$$

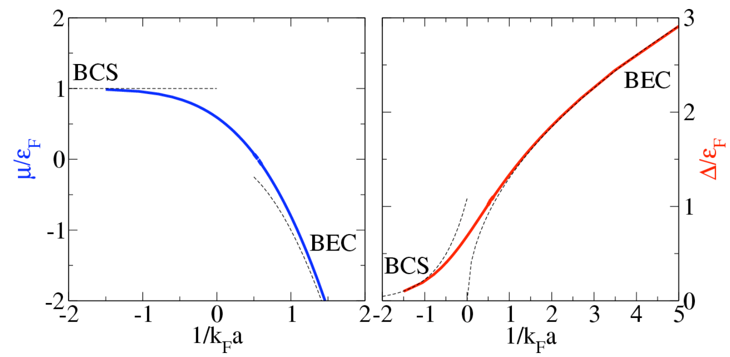
Interaction between Molecular-Molecular $a_B \approx 0.6 a_F$ **Exact**

Mean Field Method: BCS limit

BCS Limit

$$k_F a_s \rightarrow 0^-$$

$$\mu \approx E_F \qquad \Delta << \mu$$



We take cut off

$$\begin{cases} 1 = \dfrac{4\pi\hbar^2 a}{m}\displaystyle\int \dfrac{d^3\mathbf{k}}{(2\pi)^3}\left[\dfrac{1}{2\varepsilon_{\mathbf{k}}} - \dfrac{1}{2E_{\mathbf{k}}}\right], \\ n = \displaystyle\int \dfrac{d^3\mathbf{k}}{(2\pi)^3}\left(1 - \dfrac{\varepsilon_{\mathbf{k}} - \mu}{E_{\mathbf{k}}}\right). \end{cases}$$

$$\frac{\sqrt{\varepsilon_k}}{E_k} \approx \frac{1}{\sqrt{\varepsilon_k} + \sqrt{\mu'}} + \frac{\sqrt{\mu'}}{E_k}$$

$$\mu' = \mu - \frac{\Delta^2}{2(\varepsilon_k - \mu)}$$

Order Parameter

$$\Delta = \frac{8}{e^2} E_F e^{\frac{\pi}{2a_s k_F}}$$

Transition Temperature

$$\frac{m}{4\pi\hbar^2 a_s} = \int \frac{d\vec{k}}{(2\pi)^3}\left(\frac{1}{2\varepsilon_k} - \frac{1}{2E_k} th \frac{E_k}{2k_B T}\right)$$

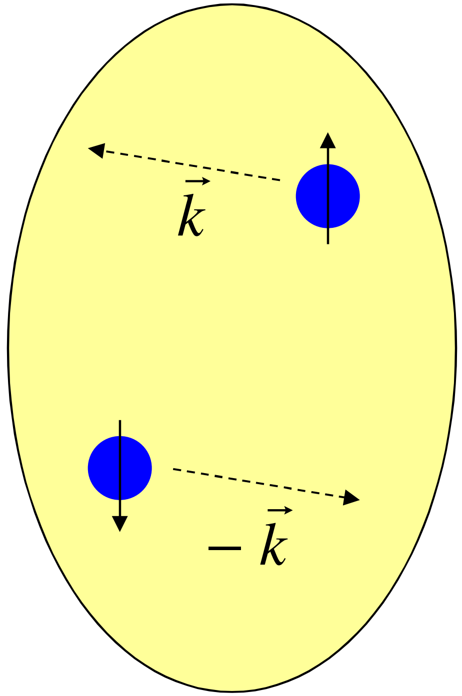
$$T_C^{BCS} = \frac{8e^{\nu}}{e^2\pi} T_F e^{\frac{\pi}{2a_s k_F}} \qquad (\nu = 0.577)$$

BCS superfluidity

Cooper pairs - BCS superfluidity

Tc~0 exponentially difficult to reach

$$T_{BCS} \approx 0.28 T_F e^{\frac{\pi}{2k_F a_s}}$$ (valid for $k_F|a|<<1$)



e.g.: $k_F a=-0.2$ -> $T_{BCS} \sim 10^{-4}\, T_F$ (very very small)

(very) low-temperature effect

Collective mode

Leggett model

$$
\begin{cases}
1 = \dfrac{4\pi\hbar^2 a}{m} \displaystyle\int \dfrac{d^3\mathbf{k}}{(2\pi)^3}\left[\dfrac{1}{2\varepsilon_{\mathbf{k}}} - \dfrac{1}{2E_{\mathbf{k}}}\right], & \textit{(gap equation)} \\[2ex]
n = \displaystyle\int \dfrac{d^3\mathbf{k}}{(2\pi)^3}\left(1 - \dfrac{\varepsilon_{\mathbf{k}} - \mu}{E_{\mathbf{k}}}\right). & \textit{(number equation)}
\end{cases}
$$

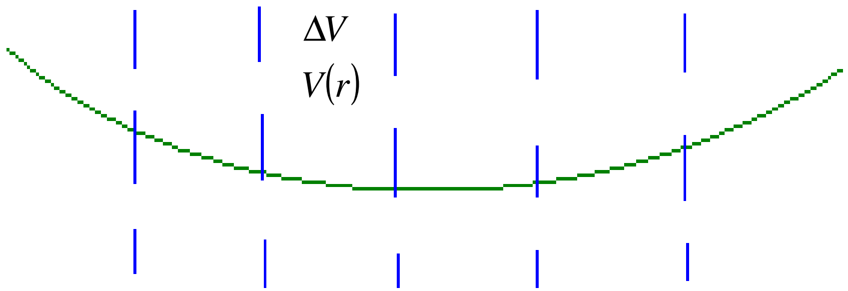
note: $E_{\mathbf{k}} = \sqrt{(\varepsilon_{\mathbf{k}} - \mu)^2 + \Delta^2}$, *and the* **Hartree** *term is not included.*

⬇

	BCS	*unitary limit*	BEC
$\mu(n)$	$\sim n^{2/3}$	$\sim n^{2/3}$	$\sim -E_b/2 + g_m n$

Local Density Approximation

fix $N\ ,T$



$\mu \leftrightarrow n$ EoS in Homogeneous System

$$\mu(r) = \mu_g - V(r)$$

$$n(\mu) = n\left(\mu_g - \frac{1}{2}m\omega^2 r^2\right)$$

μ_g : determined by N

$$N = \int d\vec{r}\, n(\vec{r})$$

Leggett model: details

$\gamma(n) = 1 + n\,[(d^2\mu/dn^2)\,/\,(d\mu/dn)]$

from Leggett model, for a fixed x.

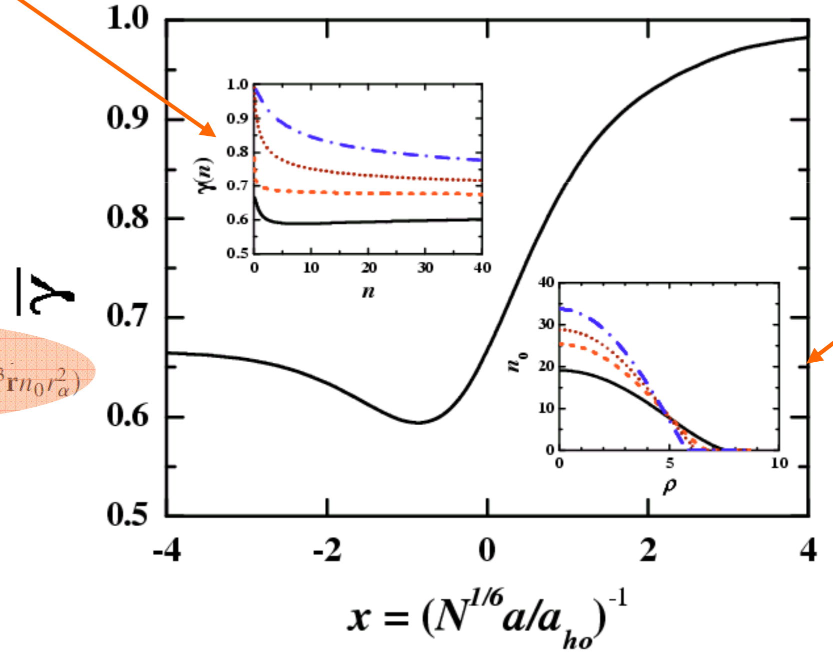
$\lambda = \omega_z / \omega_\perp = 0.05$ ⑥ 1

$\mu \propto n^{\gamma}$

$\bar{\gamma} = [\int d^3\mathbf{r}\, n_0 r_\alpha^2 \gamma(n_0)]/(\int d^3\mathbf{r}\, n_0 r_\alpha^2)$

gs. density profiles from LDA

$\mu(n_0(\mathbf{r})) + V_{ext}(\mathbf{r}) = \mu_g$



in the insets, from bottom to top, x =-1.0, 0.1, +0.5 and +1.0 (for the curves)

Calculate collective mode frequencies

superfluid hydrodynamic equations

$$\begin{cases} \partial_t n + \nabla \cdot (n\mathbf{v}) = 0, & (\textit{equation of continuity}) \\ m\partial_t \mathbf{v} + \nabla\left[\mu(n) + V_{trap} + m\mathbf{v}^2/2\right] = 0. & (\textit{Euler equation}) \end{cases}$$

due to the trap

pair phase fluctuations → *density oscillations*

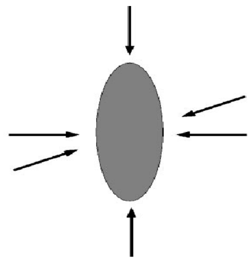
validity of hydrodynamic Eqs., see, M. A. Baranov *et al.*, PRA 62, 041601(R) (2000)

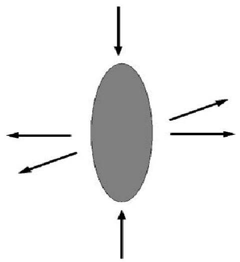
Hydrodynamic Equation

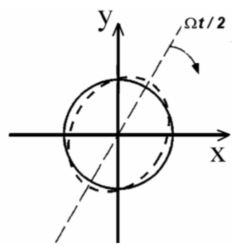
scaling solutions of hydrodynamic Eqs.

$$n(\mathbf{r},t)=\frac{n_0\left(\frac{x}{b_x(t)},\frac{y}{b_y(t)},\frac{z}{b_z(t)}\right)}{\prod_{\alpha=x,y,z} b_\alpha(t)},\qquad v_\alpha(\mathbf{r},t)=\frac{\dot{b}_\alpha(t)r_\alpha}{b_\alpha(t)},$$

$$\omega_{\perp,z}=\omega_{\perp,z}\left[1+\varepsilon\cos(\omega t)\right]$$





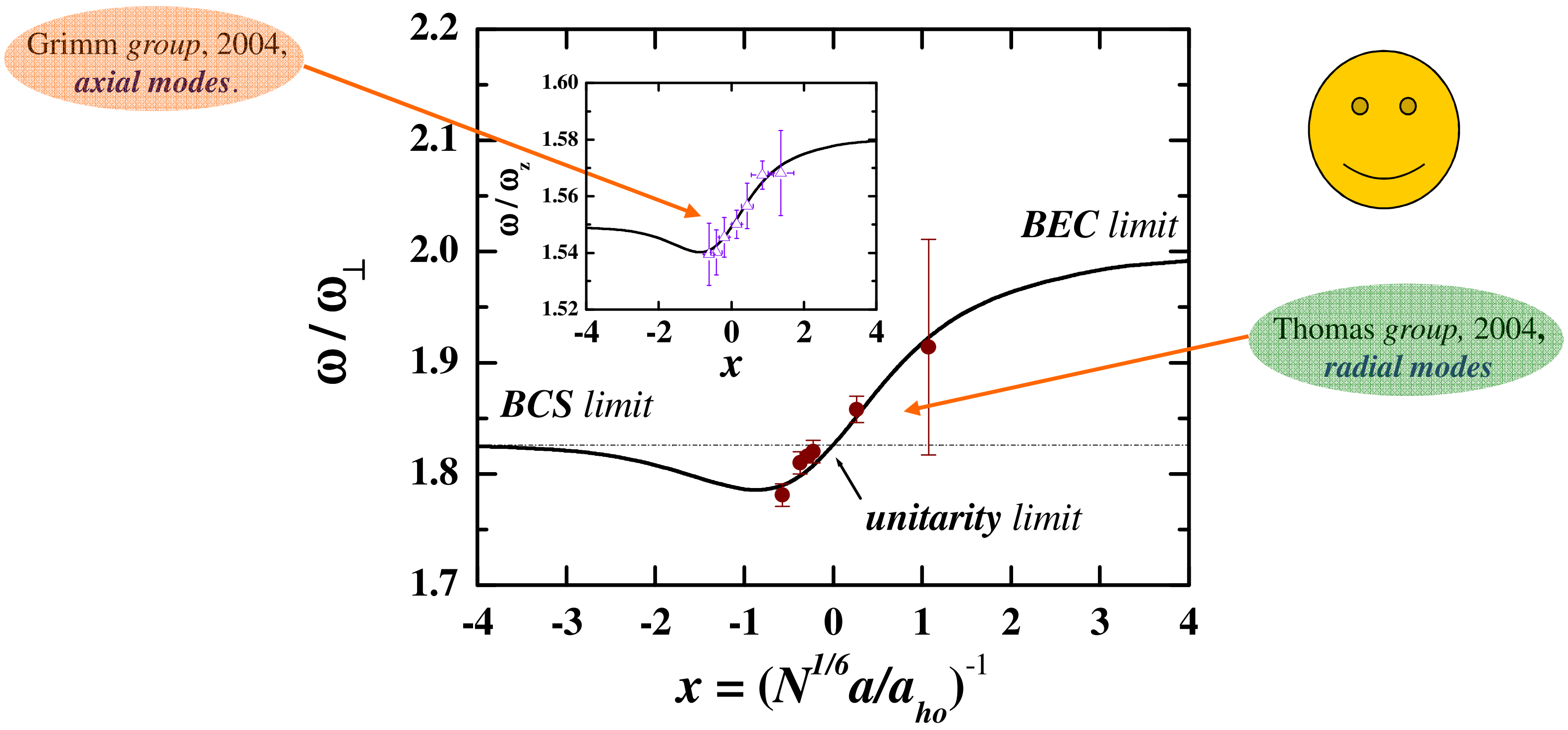


$\delta n \sim Y_{00}(\theta,\phi)$ + $\delta n \sim Y_{20}(\theta,\phi)$, $\delta n \sim Y_{22}(\theta,\phi)$ *surface mode*

$$\omega_\pm^2/\omega_\perp^2=1+\bar{\gamma}+(2+\bar{\gamma})\lambda^2/2 \pm\sqrt{[1+\bar{\gamma}+(2+\bar{\gamma})\lambda^2/2]^2-2(2+3\bar{\gamma})\lambda^2},$$

Compare with Experimental Results

BCS mean field *works well* for all accessible fields at T=0

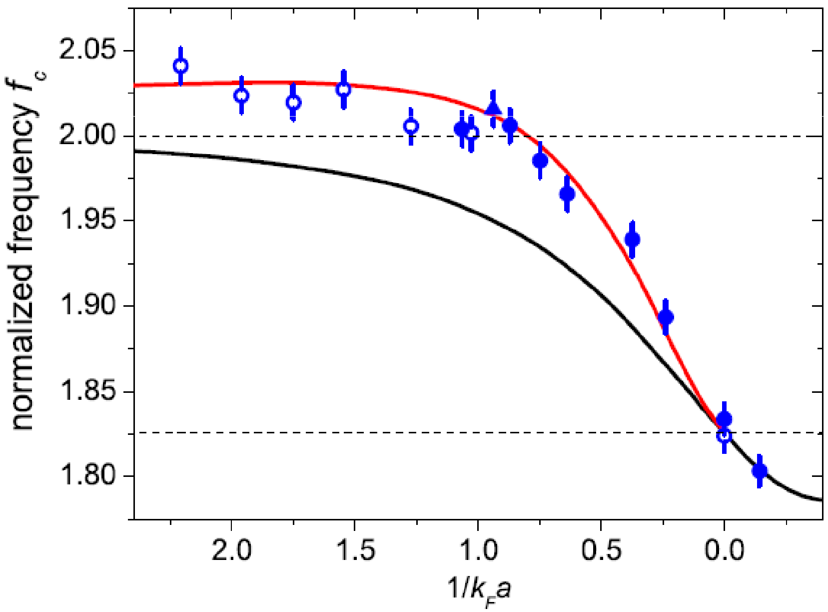


H. Hu, A. Minguzzi, X.-J. Liu, and M. P. Tosi, *Phys. Rev. Lett.* **93**, 190403 (2004).

Experimental Results in 2007

BCS-BEC crossover gas: zero temperature

BCS mean field → *well-understood, but qualitative !*



BCS-MF ☹

Rudi Grimm *et al.*, PRL 2007.

Bloch, Dalibard and Zwerger RMP Vol 80, 885(2008)

Hui Hu, Xia-Ji Liu and P. D. Drummond, EPL 74, 574(2006)

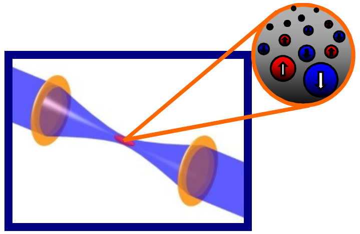
High Temperature Virial Expansion

Xia-Ji Liu

CAOUS, Swinburne University

Hawthorn, June.

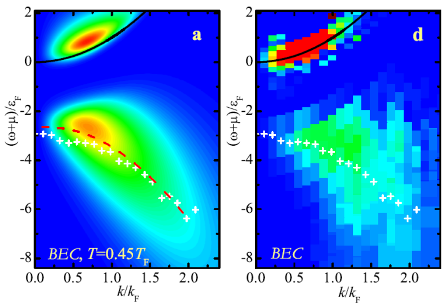
Outline



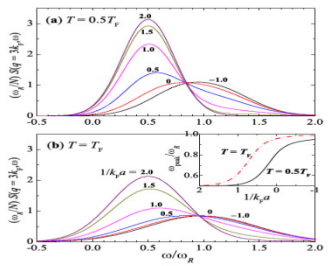
- **Virial expansion: A traditional but “new” method**
- **Few-particle exact solutions as the input to virial expansion**
- **Virial expansion: Applications**

$$b_3 = (Q_3 - Q_1 Q_2 + \frac{1}{3} Q_1^3), \ldots$$

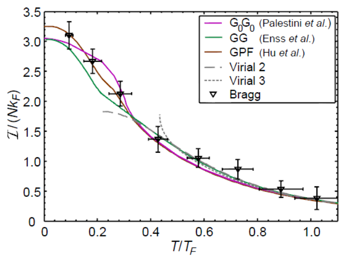
Equation of State



SP Spectral Function



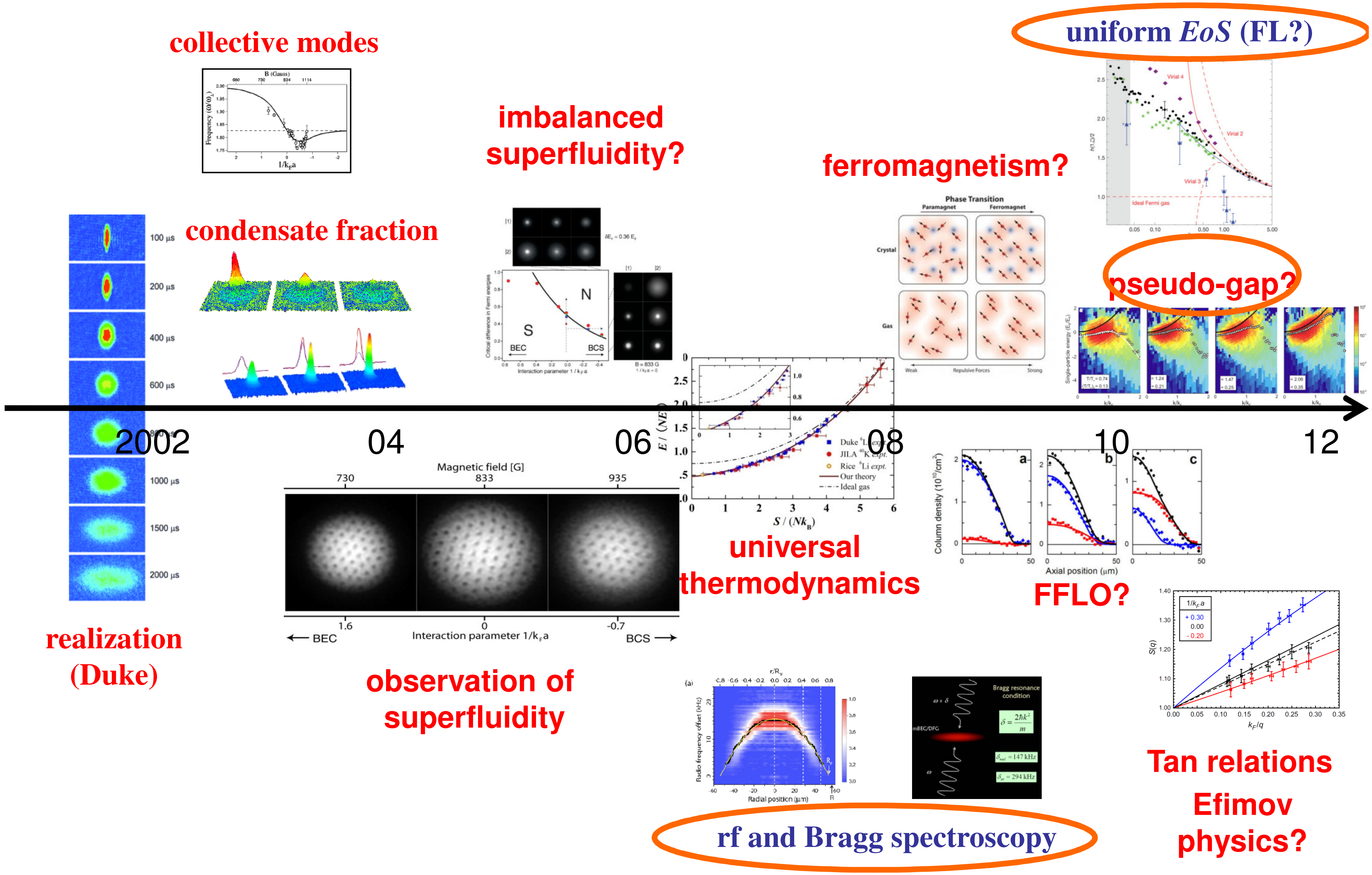
Dynamic Structure Factor



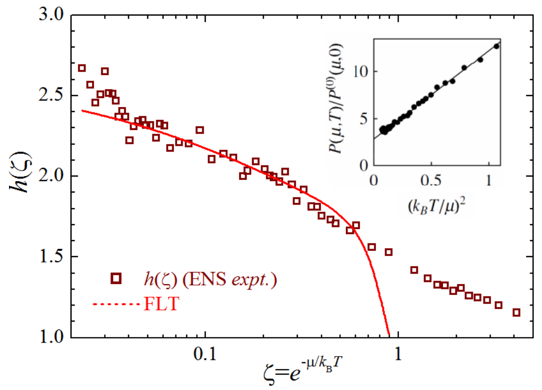
Tan’s Contact

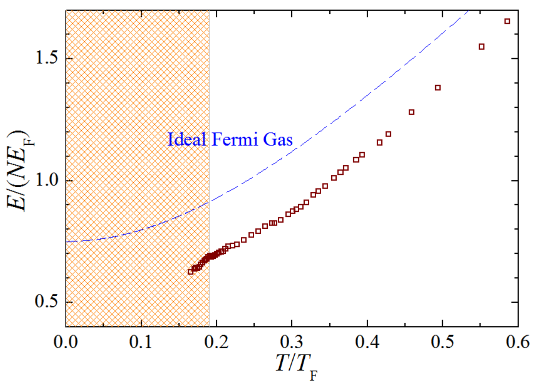
- **Conclusions and outlooks**

Global progress (experiment)

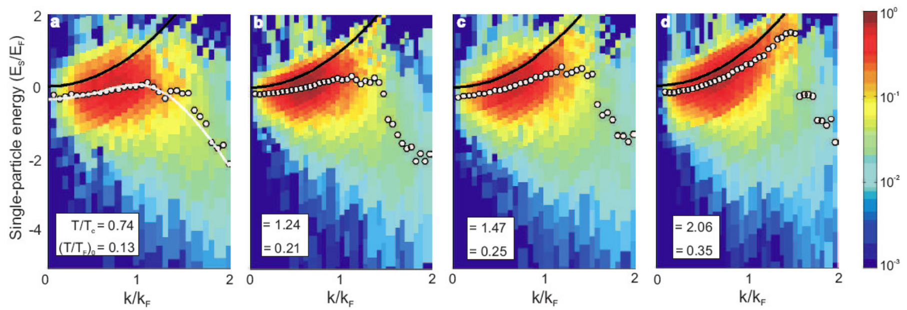


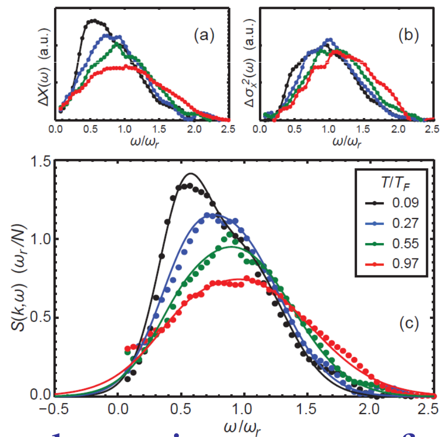
How to understand these experimental results?





unitary equation of state (static)



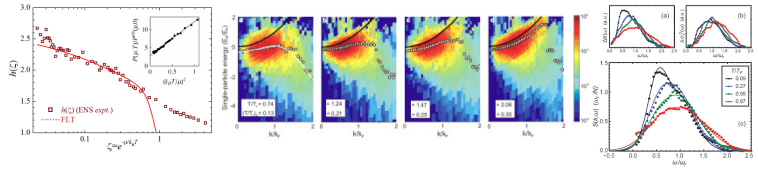


unitary single-particle spectral function

unitary dynamic structure factor

It is a central, grand challenge to theorists, due to the lack of small interaction parameter!

BEC-BCS crossover: (theoretical challenge)



☺MB physics
at high T:
Virial expansion

Few-particle solutions

Bridge?

Many-Body physics:
☹ ***No* small interaction parameter!**

☺MB physics
at short-range:
Tan relations

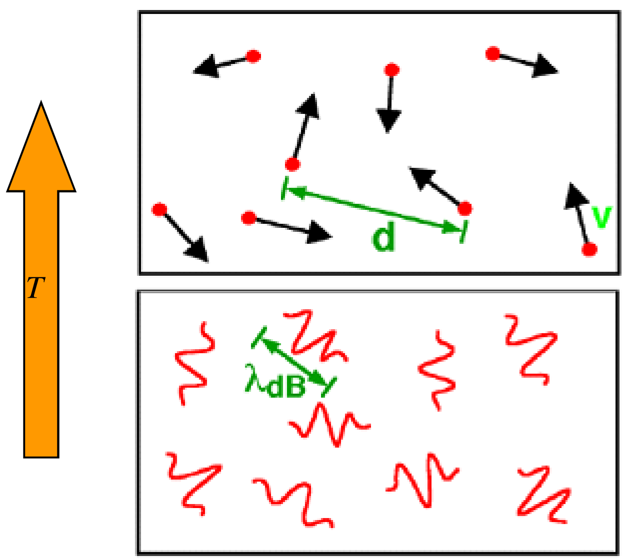
Blume, Daily, Stecher, Greene;
Busch, Englert, Rzazewski, Wilkens;
Werner, Castin;
Kestner, Duan;
Julienne, Bohn, Tiesinga;
……

i.e., $\Delta S(q,\omega) \approx W(q,\omega) C$

Virial expansion:
A traditional but "new" method

ABC of virial expansion (VE)

Classical Particles



High Temperature

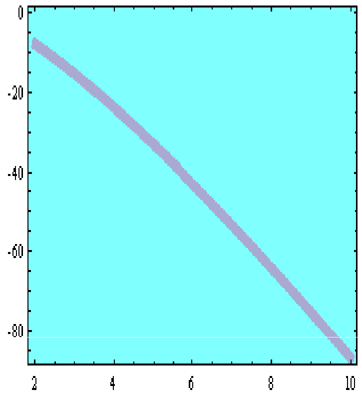
"Billiard balls"

Low Temperature

"Wave packets"

Thermal fluctuation

ABC of virial expansion (VE)



$$\frac{\mu}{E_F}$$

$$\frac{T}{T_F}$$

$$\mu(T,N) = -k_B T \ln\left[6\left(\frac{k_B T}{E_F}\right)^3\right]$$

$$\mu \to -\infty$$

The fugacity $z = \exp(\mu / k_B T) << 1$

ABC of virial expansion (VE)

Thermodynamic potential

$$\Omega(T,V,\mu) = -k_B T \ln Z_G$$

$$Z_G = Tr\left(e^{-\beta(H_0-\mu N)}\right)$$

$$Z_G = \sum_N \sum_j e^{-\beta(E_j-\mu N)}$$

$$Z_G = 1 + zQ_1 + z^2 Q_2 + z^3 Q_3 \cdots$$

***N*-cluster partition function:**

$$Q_N = Tr_N[\exp(-\beta H_N)]$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots \qquad |x| \leq 1$$

$$\Omega = -k_B T Q_1 \left(z + b_2 z^2 + b_3 z^3 + \cdots + b_n z^n + \cdots\right)$$

Virial Coefficients

$$b_2 = (Q_2 - \frac{1}{2}Q_1^2)/Q_1, \qquad b_3 = (Q_3 - Q_1 Q_2 + \frac{1}{3}Q_1^3), \qquad b_4 = \ldots$$

To obtain b_n, just solve a "n-body" problem and find out the energy levels !

b_2: T.-L. Ho & E. J. Mueller, *PRL* **92**, 160404 (2005).
b_3: Liu, HH & Drummond, *PRL* **102**, 160401 (2009); *PRA* **82**, 023619 (2010).

ABC of virial expansion (VE)

Numerically, we calculate $\Delta b_n = b_n - b_n^{(1)}$ **for a trapped gas!**

***n*-th virial coefficient of a non-interacting Fermi gas**

What's new here?

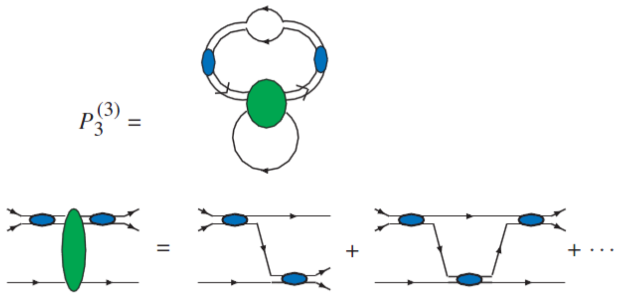
For a homogeneous system, where the energy level is continuous, it seems impossible to calculate directly virial coefficient using N-cluster partition function, *i.e.*, $b_3 = (Q_3 - Q_1 Q_2 + \frac{1}{3} Q_1^3), \ldots$

For the second virial coefficient, Beth & Uhlenbeck (1937):

$$\frac{\Delta b_2}{\sqrt{2}} = \sum_i \mathrm{e}^{-E_b^i/(k_B T)} + \frac{1}{\pi}\int_0^\infty \mathrm{d}k \frac{\mathrm{d}\delta_0}{\mathrm{d}k} \mathrm{e}^{-\lambda^2 k^2/(2\pi)}$$

δ_0: *s*-wave phase shift;
λ: de Broglie wavelength.

For the third coefficient, complicated diagrammatic calculations [Rupak, *PRL* **98**, 090403 (2007)] :



leads to, $\Delta b_3(\text{Rupak}) \approx 1.05$ (incorrect ☹)

The harmonic trap helps! The discrete energy level helps to calculate the *N*-cluster partition function.

How to obtain homogeneous virial coefficient?

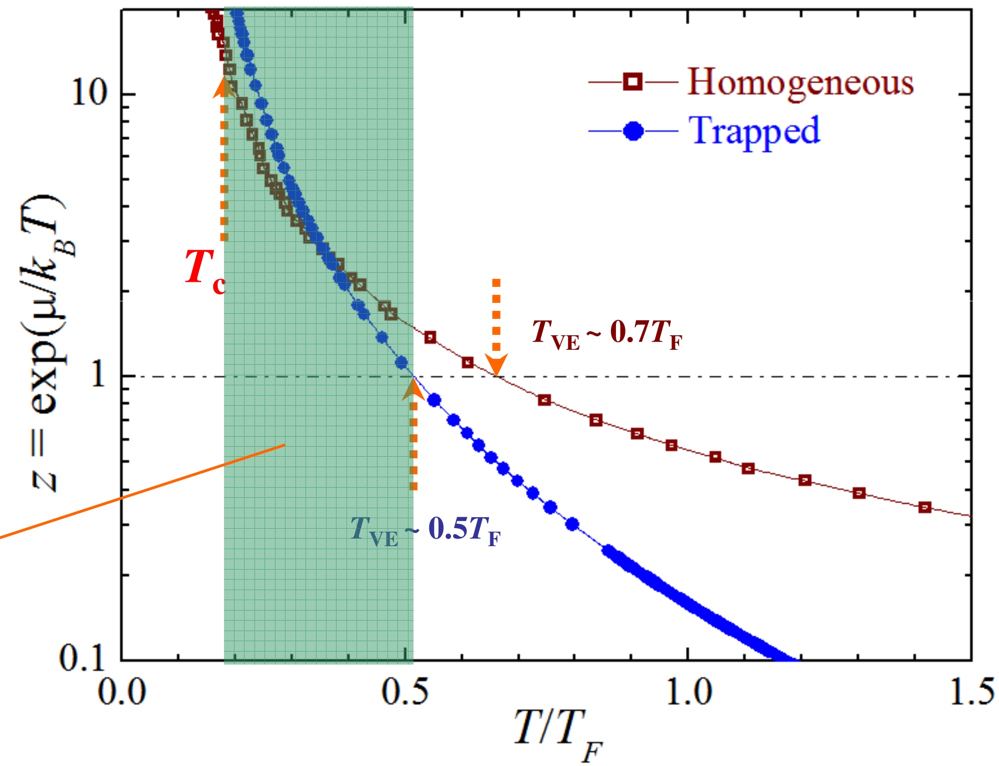
Let us consider the *unitarity* limit and use **LDA** **[μ(r) = μ - *V*(r)],**

$$\Omega_{trap} \propto \sum_{n=1} b_{n,T} z^n \overset{\textbf{LDA}}{\propto} \int d\mathbf{r} \sum_{n=1} b_{n,H} z^n(\mathbf{r}) = \int d\mathbf{r} \sum_{n=1} b_{n,H} z^n \exp[-n\beta V(\mathbf{r})]$$

$$b_{n,T}(\text{trap}) = \left[\frac{1}{n^{3/2}}\right] b_{n,H}(\text{homogeneous})$$

Liu, HH & Drummond, *PRL* **102**, 160401 (2009); *PRA* **82**, 023619 (2010).

Validity of virial expansion? (unitarity case)



Non-trivial re-summation of virial expansion terms? *i.e.*, *Páde* approximation?

Unitary ***z*(*T*)** from the ENS data; *see*, HH, Liu & Drummond, *New. J. Phys.* **12**, 063038 (2010).

Virial expansion of single-particle spectral function

$$G_{\sigma\sigma'}\left(\mathbf{r},\mathbf{r}';\tau\right) = -\exp\left[\mu\tau\right]\frac{1}{\mathcal{Z}}\mathrm{Tr}\left[z^{\mathcal{N}}e^{-\beta\mathcal{H}}e^{\tau\mathcal{H}}\hat{\Psi}_{\sigma}\left(\mathbf{r}\right)e^{-\tau\mathcal{H}}\hat{\Psi}^{+}_{\sigma'}\left(\mathbf{r}'\right)\right]$$

$$= A_1 + z\left(A_2 - A_1 Q_1\right) + \cdots,$$

virial expansion functions:

$$A_N = -\exp\left[\mu\tau\right]\mathrm{Tr}_{N-1}\left[e^{-\beta\mathcal{H}}e^{\tau\mathcal{H}}\hat{\Psi}_{\sigma}\left(\mathbf{r}\right)e^{-\tau\mathcal{H}}\hat{\Psi}^{+}_{\sigma'}\left(\mathbf{r}'\right)\right]$$

To obtain A_n, solve a "*n*-body" problem and the wave functions!

HH, Liu, Drummond & Dong, *PRL* **104**, 240407 (2010).

Quantum virial expansion of DSF

VE for dynamic susceptibility: $$\chi_{\sigma\sigma'} \equiv -\frac{\mathrm{Tr}\left[e^{-\beta(\mathcal{H}-\mu\mathcal{N})}e^{\mathcal{H}\tau}\hat{n}_{\sigma}(\mathbf{r})\,e^{-\mathcal{H}\tau}\hat{n}_{\sigma'}(\mathbf{r}')\right]}{\mathrm{Tr}e^{-\beta(\mathcal{H}-\mu\mathcal{N})}}$$

$$\chi_{\sigma\sigma'}(\mathbf{r},\mathbf{r}';\tau) = zX_1 + z^2(X_2 - X_1Q_1) + \cdots$$

virial expansion functions: $$X_n = -\,\mathrm{Tr}_n[e^{-\beta\mathcal{H}}e^{\tau\mathcal{H}}\hat{n}_{\sigma}(\mathbf{r})e^{-\tau\mathcal{H}}\hat{n}_{\sigma'}(\mathbf{r}')]$$

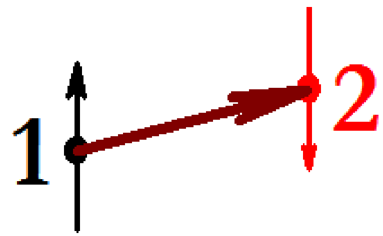
Finally, we use $$S_{\sigma\sigma'}(\mathbf{r},\mathbf{r}';\omega) = -\frac{\mathrm{Im}\,\chi_{\sigma\sigma'}(\mathbf{r},\mathbf{r}';i\omega_n\rightarrow\omega+i0^+)}{\pi(1-e^{-\beta\omega})}$$

HH, Liu, & Drummond, *PRA* **81**, 033630 (2010).

Few-particle exact solutions: As the input to virial expansion

Blume, Daily, Stecher, Greene;
Busch, Englert, Rzazewski, Wilkens;
Werner, Castin;
Kestner, Duan;
Julienne, Bohn, Tiesinga;
……

Two-particle problem in harmonic traps



CM motion: $\left[-\frac{\hbar^2}{2M}\Delta_{\vec{C}} + \frac{1}{2}M\omega^2 C^2\right]\psi_{\text{CM}}(\vec{C}) = E_{\text{CM}}\psi_{\text{CM}}(\vec{C}),$ $\boxed{E_{\text{CM}} \in (\frac{3}{2} + \mathbb{N})\hbar\omega}$

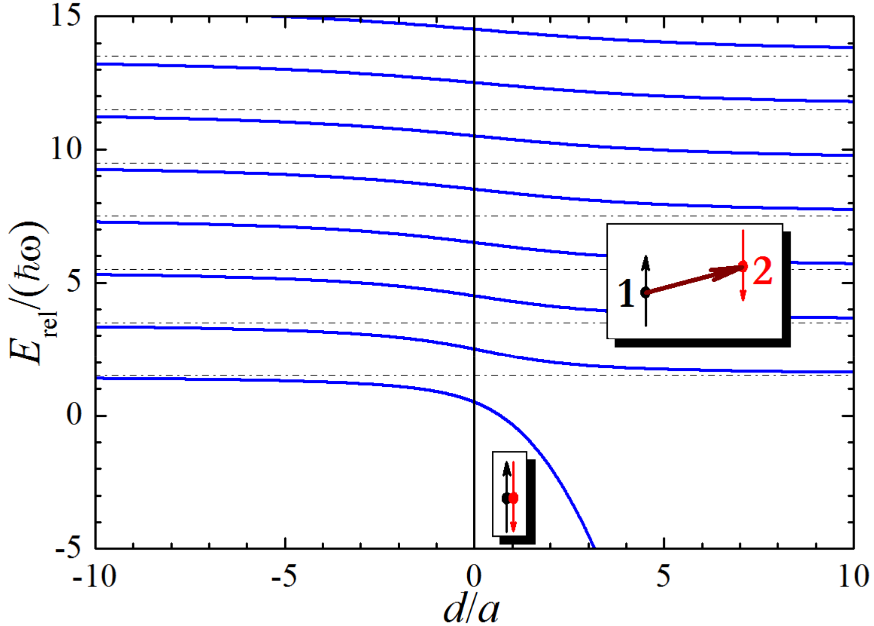
Relative motion: $\left[-\frac{\hbar^2}{2\mu}\nabla^2_{\mathbf{r}} + \frac{1}{2}\mu\omega^2 r^2\right]\psi^{\text{rel}}_{2b}(\mathbf{r}) = E_{\text{rel}}\psi^{\text{rel}}_{2b}(\mathbf{r}),$ $\boxed{\psi^{\text{rel}}_{2b}(r) \to (1/r - 1/a)}$ **BP** condition

The solution:

- $\psi^{\text{rel}}_{2b}(r;\nu) = \Gamma(-\nu)U\left(-\nu, \frac{3}{2}, \frac{r^2}{d^2}\right)\exp\left(-\frac{r^2}{2d^2}\right)$
- U is the second Kummer function
- $E_{\text{rel}} = \left(2\nu + \frac{3}{2}\right)\hbar\omega$ is determined from the BP condition

See, Busch *et al.*, *Found. Phys.* (1998)

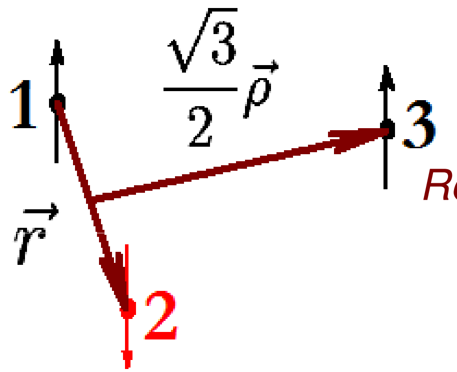
Two-particle problem in harmonic traps



Analytic result is known at unitarity: $E_{\rm rel} = \left(2n + \frac{1}{2}\right)\hbar\omega, \ n \in \mathbb{N}.$ [*See*, Busch *et al.*, *Found. Phys.* (1998)]

$$b_2 - b_2^{(1)} = (Q_2 - Q_2^{(1)})/Q_1 = \frac{1}{2}\left[\sum_n \exp(-\beta E_{rel,n}) - \sum_n \exp(-\beta E_{rel,n}^{(1)})\right] = \left(\frac{1}{4}\right)\frac{2\exp(-\beta\hbar\omega/2)}{1+\exp(-\beta\hbar\omega)},$$

Three-particle problem in harmonic traps



CM motion: $\left[-\frac{\hbar^2}{2M}\Delta_{\vec{C}}+\frac{1}{2}M\omega^2C^2\right]\psi_{\rm CM}(\vec{C})=E_{\rm CM}\psi_{\rm CM}(\vec{C}),\ \boxed{E_{\rm CM}\in(\frac{3}{2}+\mathbb{N})\hbar\omega}$

Relative motion: $\left[-\frac{\hbar^2}{m}\left(\Delta_{\vec{r}}+\Delta_{\vec{\rho}}\right)+\frac{1}{4}m\omega^2(r^2+\rho^2)\right]\psi(\vec{r},\vec{\rho})=E\,\psi(\vec{r},\vec{\rho})$

BP condition: $\psi(\vec{r},\vec{\rho})\underset{r\to 0}{=}\left(\frac{1}{r}-\frac{1}{a}\right)A(\vec{\rho})+O(r)$

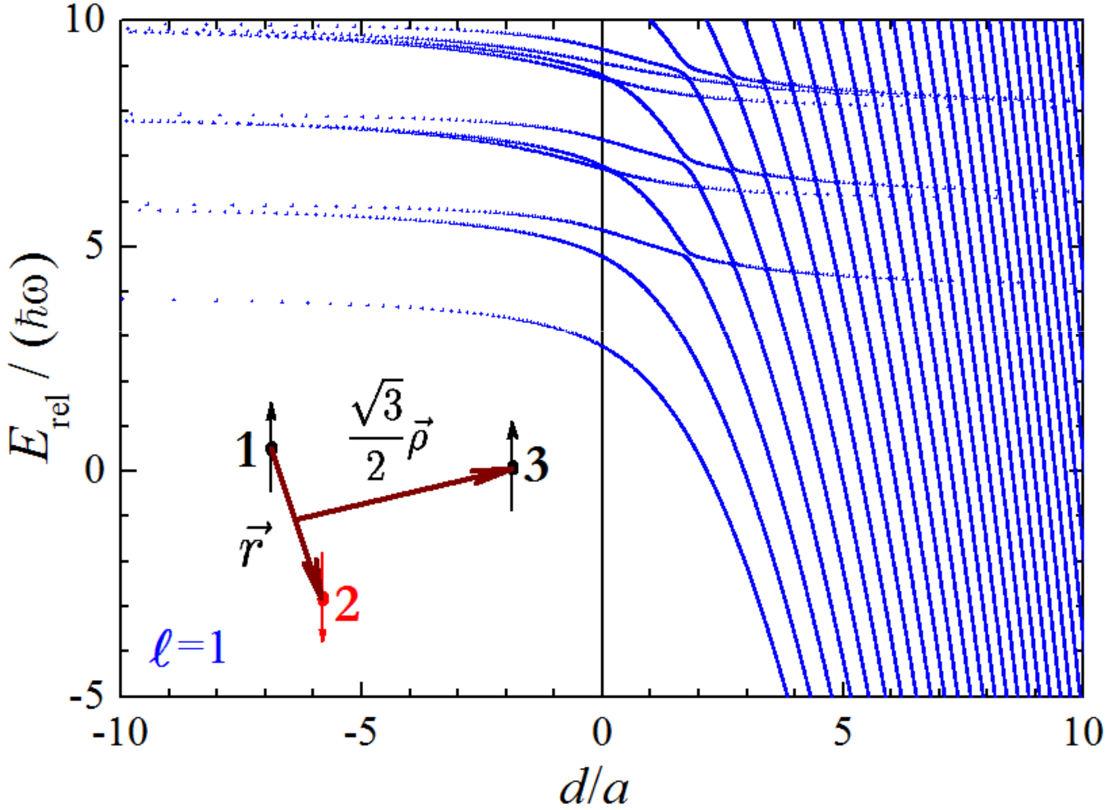
In general: $\psi(\vec{r},\vec{\rho})=(\hat{\mathbf{1}}-\hat{\mathbf{P}}_{13})\sum_n a_n\phi_{nl}(\rho)Y_{lm}(\hat{\rho})\Gamma(-\nu_n)U(-\nu_n,\frac{3}{2};r^2)\exp(-\frac{r^2}{2})Y_{00}(\hat{r})$

(P_{13}: particle exchange operator)

$[(2n+l+\frac{3}{2})+(2\nu_n+\frac{3}{2})]\hbar\omega=E_{rel}$
is determined from the BP condition

Liu, HH & Drummond, *PRA* **82**, 023619 (2010)

Three-particle problem in harmonic traps



Relative energy levels "*E*" as a function of the inverse scattering length (l = 1 section).

Three-particle problem at unitarity

$$R = \sqrt{\frac{r^2+\rho^2}{2}}, \quad \vec{\Omega} = (\alpha, \hat{r}, \hat{\rho})$$

$$\alpha = \arctan\left(\frac{r}{\rho}\right)$$

Separable wavefunctions !

$$\psi(R,\vec{\Omega}) = \frac{F(R)}{R^2}(1-\hat{P}_{13})\frac{\varphi(\alpha)}{\sin(2\alpha)}Y_l^m(\hat{\rho})$$

(*P_{13}: particle exchange operator*)

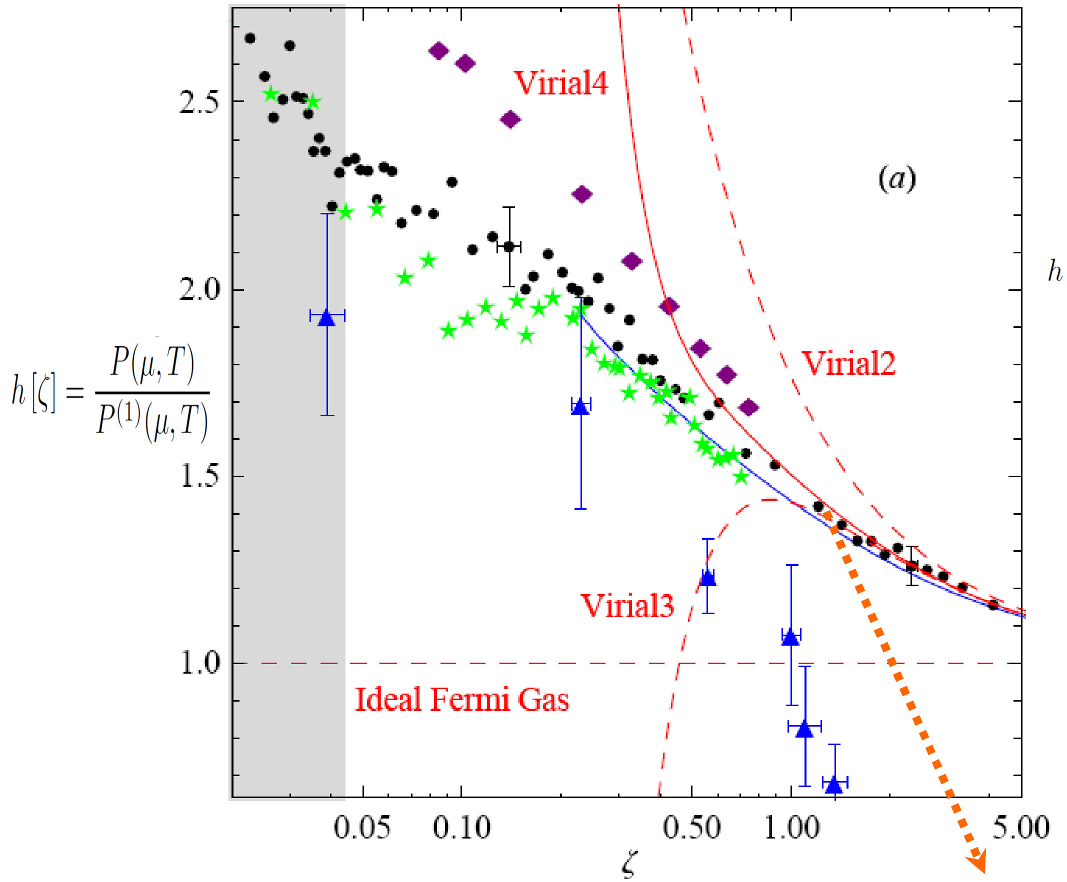
See, Werner & Castin, *PRL* (2006): $$E_{rel} = 1 + 2q + s_{\ln}$$

$$b_3 - b_3^{(1)} = \frac{Q_3 - Q_3^{(1)}}{Q_1} - (Q_2 - Q_2^{(1)}) = \frac{e^{-\beta\hbar\omega}}{1-e^{-2\beta\hbar\omega}}\sum_{l,n}(2l+1)\left[\exp(-\beta\hbar\omega s_{\ln}) - \exp(-\beta\hbar\omega s_{\ln}^{(1)})\right],$$

Numerically, $$b_3 - b_3^{(1)} = \mathbf{-0.06833960} + \mathbf{0.038867}\left(\frac{\hbar\omega}{k_B T}\right)^2 - \mathbf{0.0135}\left(\frac{\hbar\omega}{k_B T}\right)^4 + \ldots,$$

Virial expansion: Applications

Virial coefficient at unitarity (uniform case)



We now comment the main features of the equation of state. At high temperature, the EOS can be expanded in powers of ζ^{-1} as a virial expansion [11]:

$$h[\zeta]=\frac{P(\mu,T)}{P^{(1)}(\mu,T)}=\frac{\sum_{k=1}^{\infty}\left((-1)^{k+1}k^{-5/2}+b_k\right)\zeta^{-k}}{\sum_{k=1}^{\infty}(-1)^{k+1}k^{-5/2}\zeta^{-k}},$$

where b_k is the k^{th} virial coefficient. Since we have $b_2=1/\sqrt{2}$ in the measurement scheme described above, our data provides for the first time the experimental values of b_3 and b_4. $b_3=-0.35(2)$ is in excellent agreement with the recent calculation $b_3=-0.291-3^{-5/2}=-0.355$ from [11] but not with $b_3=1.05$ from [12]. $b_4=0.096(15)$ involves the 4-fermion problem at unitarity and could interestingly be computed along the lines of [11].

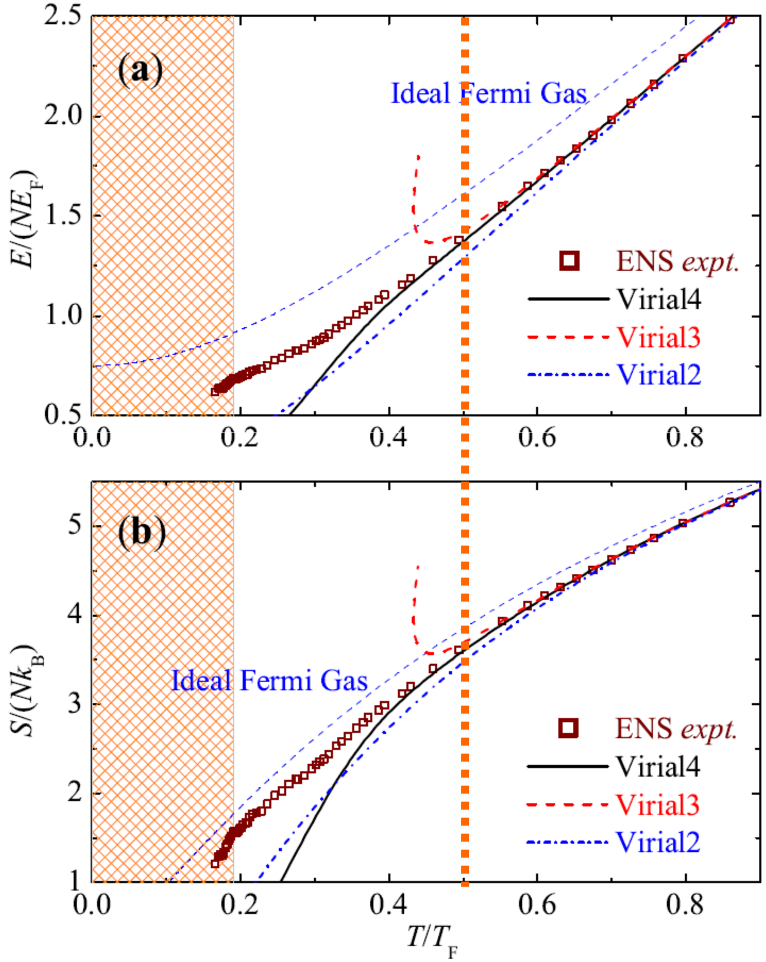
Nascimbène *et al.*, *Nature*, 25 February 2010.

$\Delta b_2=1/\sqrt{2}$ (known 70s ago)

√ Δb_3(Liu *et al.*) ≈ -0.35510298 (*PRL* 2009)

X Δb_3(Rupak) ≈ 1.05 (*PRL* 2007)

Unitary *EoS* at high *T*: trapped case



Here,

$\Delta b_2 = 1/\sqrt{2}$

$\Delta b_3 \approx -0.35510298$

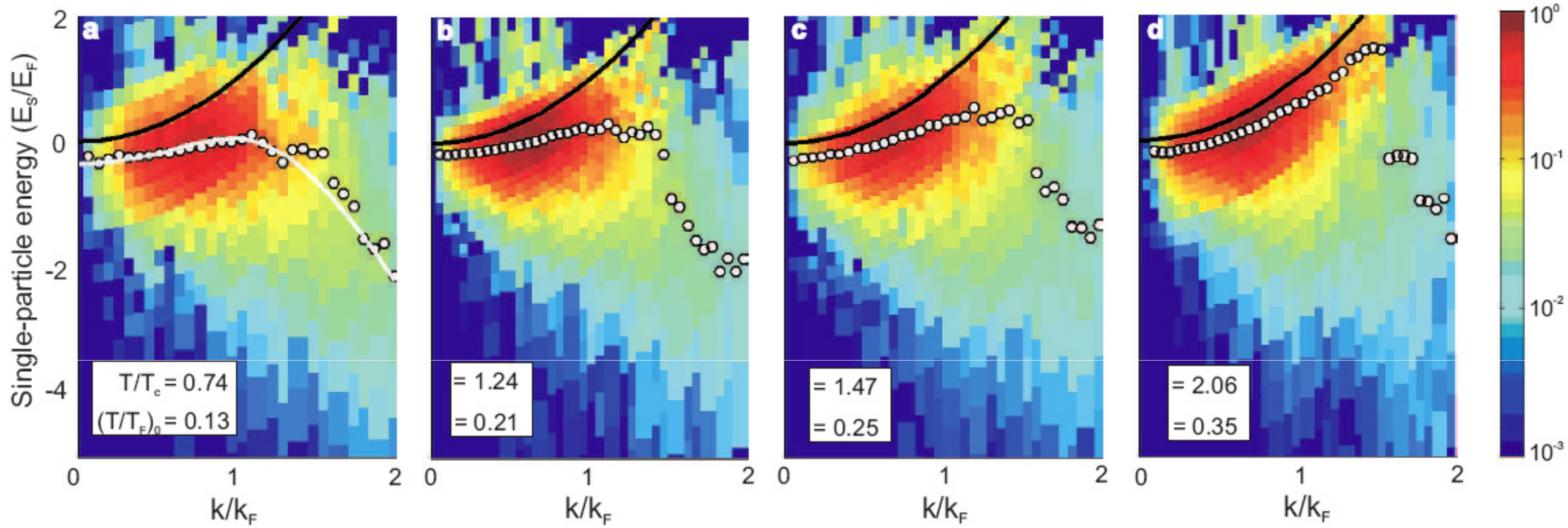
$\Delta b_4(\text{ENS}) \approx 0.096(15)$

Expt. data:
Calculated from $h(\zeta)$ of ENS's *Unitarity EoS*

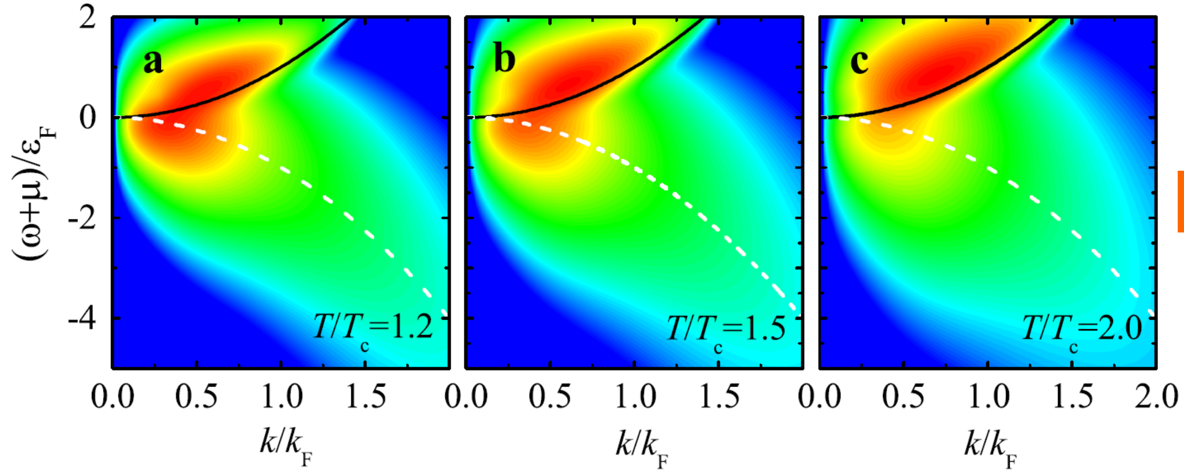
Theory data:
HH *et al.*, *New J. Phys.* **12**, 063038 (2010).

Trapped spectral function (second order only)

$$A(k,\omega) = A^{(1)}(k,\omega) + z^2 A_2(k,\omega) + \cdots$$



Expt.: JILA,
Nature Physics (2010).



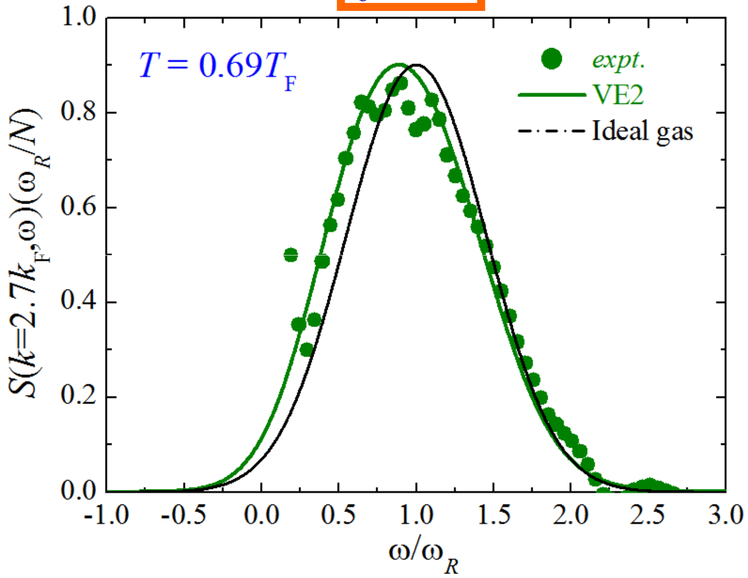
$z \sim 10$

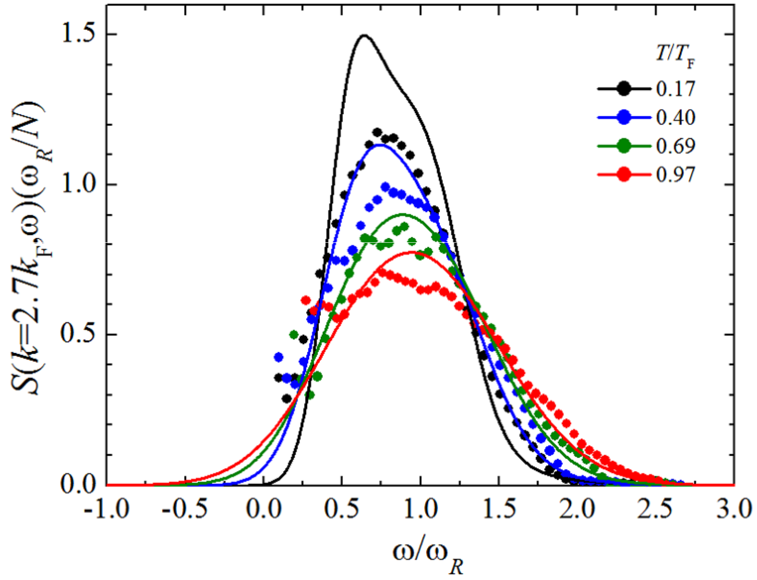
Theory: HH *et al.*,
PRL **104**, 240407 (2010).

Trapped dynamic structure factor (second order only)

$$S(k,\omega) = S^{(1)}(k,\omega) + z^2 S_2(k,\omega) + \cdots$$

$z \sim 0.4$





Expt.: Kuhnle, Hoinka, Dyke, HH, Hannaford & Vale, *PRL*, **106** 170402 (2011).

Theory: HH, Liu, & Drummond, *PRA* **81**, 033630 (2010).

VE applications (Tan's contact)

The finite-T contact may be calculated using adiabatic relation: $\left[\frac{\partial \Omega}{\partial a_s^{-1}}\right]_{T,\mu} = -\frac{\hbar^2}{4\pi m}\,\mathcal{I}$

(**high-*T* regime**) Recall that the virial expansion for thermodynamic potential,

$$\Omega = \Omega^{(1)} - \frac{2k_B T}{\lambda_{dB}^3}\left[\Delta b_2 z^2 + \Delta b_3 z^3 + \cdots\right]$$

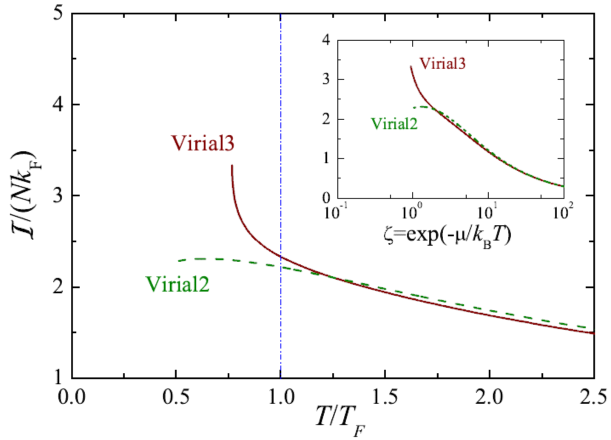
Using the adiabatic relation, it is easy to see that,

$$\mathcal{I}_{\text{virial}} = \frac{4\pi m}{\hbar^2}\frac{2k_B T}{\lambda_{dB}^2}\left[\frac{\partial \Delta b_2}{\partial(\lambda_{dB}/a_s)} z^2 + \frac{\partial \Delta b_3}{\partial(\lambda_{dB}/a_s)} z^3 + \cdots\right]$$

c_2 c_3

At the unitarity limit, we find that, $c_2 = 1/\pi$ and $c_3 \approx -0.141$. **☺ to be used as a benchmark!**

$\mathcal{I}_{\text{virial}}$

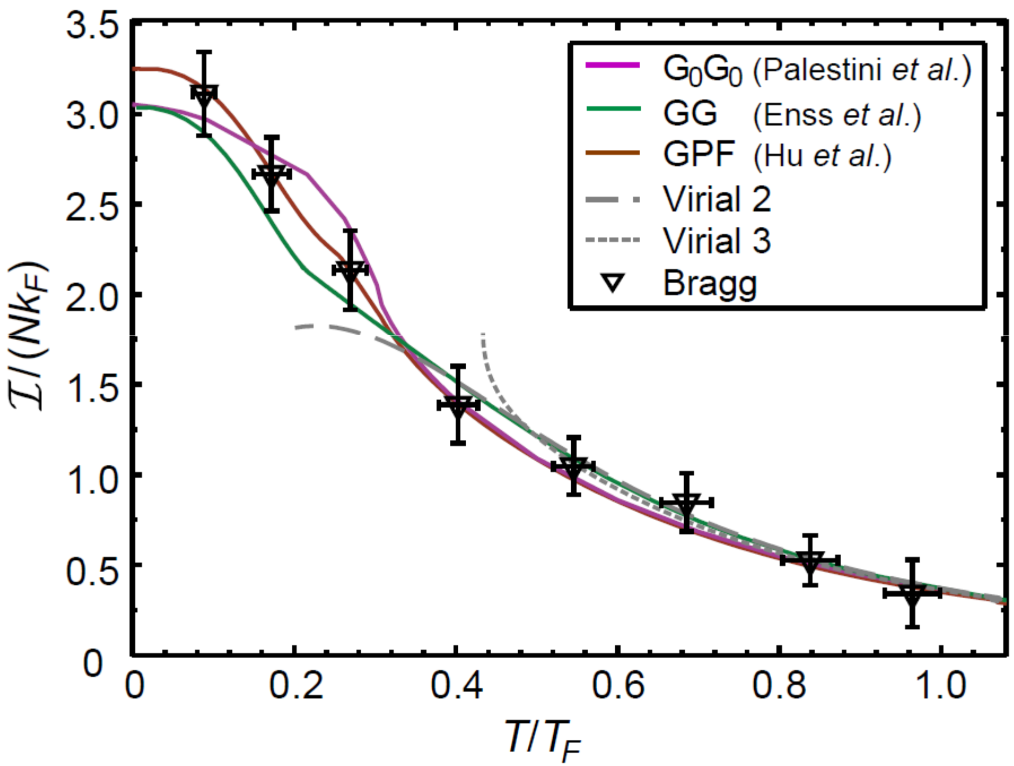


Note that,

c_n(trap)=$(1/n^{3/2})c_n$(homo)

Hu, Liu & Drummond, *NJP* **13**, 035007(2011).

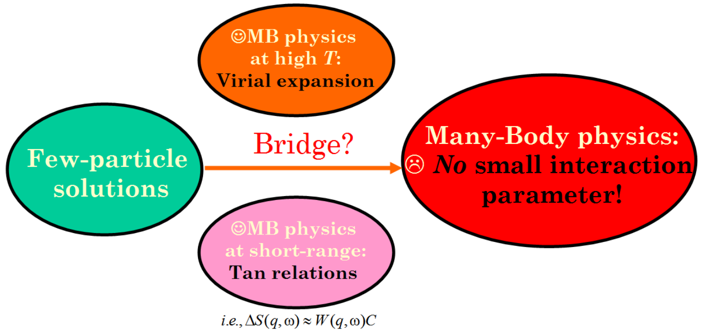
Trapped contact at unitarity (theory vs experiment)



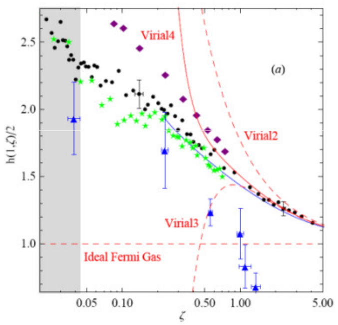
Expt.: Kuhnle, Hoinka, Dyke, HH, Hannaford & Vale, *PRL*, **106** 170402 (2011).

Theory: HH, Liu & Drummond, arXiv:1011.3845; *NJP* (2011).

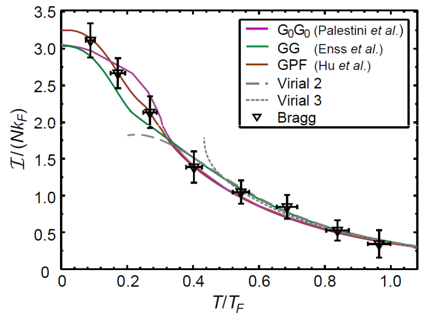
Taking home messages



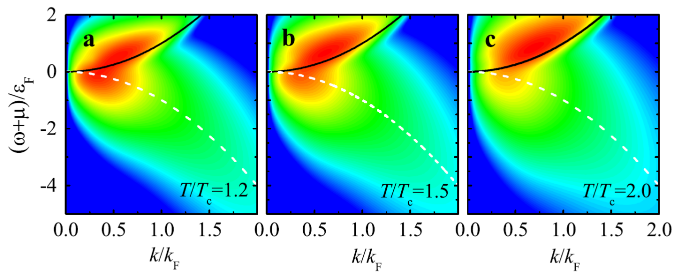
Virial expansion solves completely the large-*T* strong-correlated problem!



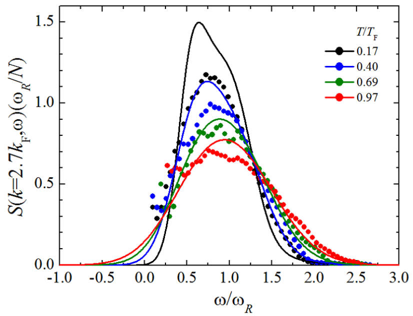
EoS



Tan's contact



SP Spectral Function



DSF

Outlooks (improved virial expansion)

- **4th order virial coefficient: experiment** $\Delta b_4 \approx 0.096$ **and theory** $\Delta b_4 \approx -0.016$

- **Can we improve *A*(*k*,ω) and *S*(*k*,ω) to the 3rd and 4th order?**

  i.e., **based on the 3- and 4-body solutions by**

  Daily & Blume;
  Stecher & Greene;
  Werner & Castin;
  ……

- **Efimov physics or *triplet* pairing response in *A*(*k*,ω) and *S*(*k*,ω) ?**